R_1 may be any one of the roots. This implies that if the roots, in the order in which they circulate, are

$$R_1,\ R_\lambda,\ R_\alpha,\ \ldots\ldots,\ R_\delta,\ R_\epsilon,\ R_\theta,$$

the change of $R_1^{\frac{1}{n}}$ in the system of equations (141) into $R_\lambda^{\frac{1}{n}}$ will cause $R_\lambda^{\frac{1}{n}}$ to become $R_\alpha^{\frac{1}{n}}$, and $R_\alpha^{\frac{1}{n}}$ to become $R_\beta^{\frac{1}{n}}$, and so on. In fact, by exactly the same reasoning as that used in establishing the Criterion of pure uni-serial Abelianism, it can be made to appear that the n values of the expression (138) or of (140) obtained by taking the n values of $R_1^{\frac{1}{n}}$ for a given value of R_1, and taking at the same time the appropriate values of $R_2^{\frac{1}{n}}$, $R_3^{\frac{1}{n}}$, etc., as determined by the equations (141), would not be the roots of an equation of the n^{th} degree with rational coefficients unless $R_\lambda^{\frac{1}{n}}$ could replace $R_1^{\frac{1}{n}}$ in the manner above indicated. In like manner, by changing $R_1^{\frac{1}{n}}$ in the system of equations (141) into $R_\alpha^{\frac{1}{n}}$, $R_\lambda^{\frac{1}{n}}$ becomes $R_\beta^{\frac{1}{n}}$, and so on. The principle can be extended to all the terms in the series

$$R_1^{\frac{1}{n}},\ R_\lambda^{\frac{1}{n}},\ R_\alpha^{\frac{1}{n}},\ \ldots\ldots,\ R_\epsilon^{\frac{1}{n}},\ R_\theta^{\frac{1}{n}}. \tag{146}$$

§ 63. Let, then, the system of equations (141) be written

$$R_{e\lambda}^{\frac{1}{n}} = a_e' R_e^{\frac{\lambda}{n}},\ R_{e\alpha}^{\frac{1}{n}} = b_e' R_e^{\frac{\alpha}{n}},\ \text{etc.}, \tag{147}$$

e being a general symbol under which all the terms in the series (143) are contained, while a_e', b_e', etc., are rational functions of R_e. These equations give us

$$(R_e^\theta R_{e\lambda}^\epsilon R_{e\alpha}^\delta \ldots\ldots R_{e\delta}^\alpha R_{e\epsilon}^\lambda R_{e\theta})^{\frac{1}{n}} = G_e R_e^{\frac{t}{n}},$$

where G_e is a rational function of R_e, and

$$t = \theta + \epsilon\lambda + \delta\alpha + \ldots\ldots + \theta = (n-1)\theta = (n-1)\lambda^{n-2}.$$

Because λ is a prime root of n, $(n-1)\lambda^{n-2}$ is prime to n. Therefore t is prime to n. Therefore whole numbers h and k exist such that

$$ht = kn + 1.$$

Therefore
$$(R_e^\theta R_{e\lambda}^\epsilon \ldots\ldots R_{e\theta})^{\frac{h}{n}} = (G_e^h R_e^k)\, R_e^{\frac{1}{n}}.$$

For every integral value of z, let $(R_{ez}^h)^{\frac{1}{n}}$ be written $r_{ez}^{\frac{1}{n}}$. Then, putting A_e^{-1} for $G_e^h R_e^k$,

$$R_e^{\frac{1}{n}} = A_e (r_e^\theta r_{e\lambda}^\epsilon r_{e\alpha}^\delta \ldots\ldots r_\delta^\alpha r_\epsilon^\lambda r_\theta)^{\frac{1}{n}}. \tag{148}$$