The latter form being obtained from the former by multiplying numerator and denominator by $-b\mp\sqrt{b^2-4ac}$, and if any of the values is indeterminate under one form, it can always be found from the other; it can also very readily be proved in any equation that if n of the coefficients of the highest powers are separately $=0$, there are n infinite roots; and if there are n coefficients of the lowest power of x including 0, separately $=0$, then there are n roots $=0$.

85. Solve the equations $x^3+y^3=35$,
$x^2+y^2=13$,
giving the six values of x and y.

$$(x+y)^3-3\,xy\,(x+y)=35 \qquad (1)$$
$$(x+y)^2-2\,xy=13 \qquad (2)$$

whence eliminating xy we have $(x+y)^3-39(x+y)+70=0$; whence by inspection $x+y=2$; $x+y=5$, and $x+y=-7$; whence from

(2) $4\,xy=-18,\ 24,$ or 72;
$x-y=\sqrt{22}$, 1, or $\sqrt{-23}$.

$$x=1\pm\tfrac{1}{2}\sqrt{22},\ 3,\ 2 \text{ and } \tfrac{1}{2}(-7\pm\sqrt{-23})$$
$$y=1\mp\tfrac{1}{2}\sqrt{22},\ 2,\ 3 \text{ and } \tfrac{1}{2}(-7\mp\sqrt{-23})$$

86. Solve $\frac{11}{x-3}-\frac{39}{x-4}+\frac{45}{x-5}-\frac{17}{x-6}=0$,
giving the three values of x.

$$-\frac{11}{x-3}+\frac{17}{x-6}=-\frac{39}{x-4}+\frac{45}{x-5}$$
$$\frac{6x+15}{x^2-9x+18}=\frac{6x+15}{x^2-9x+20}$$
$$(6x+15)\left\{x^2-9x+20-(x^2-9x+18)\right\}=0.$$
$$2\,(6x+15)=6\,(2x+5)=0.$$

when $x=-\frac{5}{2}$; the other factor is of the form $(0)\,x^2+(0)\,x+6=0$, and both values of x are infinite.

87. Prove that
$(ax+by+cz)^3+(bx+cy+az)^3+(cx+ay+bz)^3$
$=3\,(ax+by+cz)\,(bx+cy+az)\,(cx+ay+bz)$,
if $a+b+c=0$, or if $x+y+z=0$.

If $\alpha+\beta+\gamma=0$, then $\alpha+\beta=-\gamma$.
$\alpha^3+\beta^3+3\alpha\beta(\alpha+\beta)=-\gamma^3$; but $\alpha+\beta=-\gamma$
$$\therefore\ \alpha^3+\beta^3+\gamma^3=3\,\alpha\,\beta\,\gamma.$$

now $ax+by+cz=\alpha$
$bx+cy+az=\beta$
$cx+ay+bz=\gamma$

$(a+b+c)(x+y+z)=\alpha+\beta+\gamma$, and is 0, if either $a+b+c=0$, or $x+y+z=0$.

88. If $x+y+z=2s$, then will
$(s-x)^3+(s-y)^3+(s-z)^3+3\,xyz=s^3$.
$(s-x)+(s-y)+(s-z)=s$, whence cubing this equation, $(s-x)^3+(s-y)^3+(s-z)^3+3x(s-x)^2+3y(s-y)^2+3z(s-z)^2+6(s-x)(s-y)(s-z)=s^3=(s-x)^3+(s-y)^3+(s-z)^3+3(x+y+z)s^2-6s(x^2+y^2+z^2)+3(x^3+y^3+z^3)+6s^3-6(x+y+z)s^2+6s(xy+yz+zx)-6\,xyz=(s-x)^3+(s-y)^3+(s-z)^3+6s^3-6s(x^2+y^2+z^2-xy-xz-yz)+3(x^3y^3z^3)+6s^3-12s^3-6xyz=(s-x)^3+(s-y)3+(s-z)^3-3(x^3+y^3+z^3-3xyz)+3(x^3+y^3+z^3)-6xyz=(s-x)^3+(s-y)^3+(s-z)^3+3xyz=s^3$.

89. If $x+y+z=0$, then will
$$2(x^9+y^9+z^9)-9xyz(x^6+y^6+z^6)+21x^3y^3z^3=0,$$
and also
$$x^9+y^9+z^9+9x^4y^4z^4(x^{-3}+y^{-3}+z^{-3})-30x^3y^3z^3=0.$$

$x^3+y^3+z^3=3\,xyz$ (1) by cubing $x+y+z=0$.
Cubing this expression again we have,
$x^9+y^9+z^9+3x^6\,(3\,xyz-x^3)+3y^6\,(3\,xyz-y^3)+3\,z^6\,(3\,xyz-z^3)+6\,x^3y^3z^3=27\,x^3y^3z^3-2\,(x^9+y^9+z^9)+9\,xyz\,(x^6+y^6+z^6)=21\,x^3y^3z^3$, or $2\,(x^9+y^9+z^9)-9\,xyz\,(x^6+y^6+z^6)+21\,x^3y^3z^3=0$.

Second part, squaring equation (1) we have
$$x^6+y^6+z^3+2\,x^3y^3+2\,x^3z^3+2\,y^3z^3=9\,x^2y^2z^2$$
$$x^6+y^6+z^6=9x^2y^2z^2-2x^3y^3z^3(x^{-3}+y^{-3}+z^{-3})$$
$$2\,(x^9+y^9+z^9)-81\,x^3y^3z^3+18\,x^4y^4z^4\,(x^{-3}y^{-3}z^{-3})+21\,x^3y^3z^3=0.$$
$$x^9+y^9+z^9+9\,x^4y^4z^4(x^{-3}y^{-3}z^{-3})-30\,x^3y^3z^3=0.$$

90. If $\frac{b^2+c^2-a^2}{2bc}$ always lies between ±1, then will the sum of any two of these quantities be greater than the third.

$$b^2+c^2-a^2<2\,bc \text{ and } >-2\,bc$$
$$b^2+2\,bc+c^2>a^2 \qquad (1)$$
$$b^2-2\,bc+c^2<a^2 \qquad (2)$$