The Pure Uni-Serial Abelian of a Degree which is the Continued Product of a Number of Distinct Prime Numbers.

Fundamental Element of the Root.

§40. Let n be the continued product of the distinct prime numbers

$$s,\ t,\ \dots\dots,\ d,\ b. \tag{67}$$

Take w a primitive n^{th} root of unity. Then, if

$$\sigma,\ \tau,\ \dots\dots,\ \delta,\ \beta \tag{68}$$

be such that $n = s\sigma = t\tau = \dots\dots = b\beta$, w^σ is a primitive s^{th} root of unity, w^τ a primitive t^{th} root of unity, and so on. Let

$$\left.\begin{array}{l} w^\sigma,\ w^{\sigma\lambda},\ w^{\sigma\lambda^2},\ \dots\dots,\ w^{\sigma\lambda^{s-2}} \\ w^\tau,\ w^{\tau h},\ w^{\tau h^2},\ \dots\dots,\ w^{\tau h^{t-2}} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ w^\delta,\ w^{\delta l},\ w^{\delta l^2},\ \dots\dots,\ w^{\delta l^{d-2}} \\ w^\beta,\ w^{\beta k},\ w^{\beta k^2},\ \dots\dots,\ w^{\beta k^{b-2}} \end{array}\right\} \tag{69}$$

be cycles containing respectively all the primitive s^{th} roots of unity, all the primitive t^{th} roots of unity, and so on. Should the numbers forming the series (67) be all odd, each of the cycles (69) consists of more terms than one. Should the prime number 2 be a term in (67), say b, the last of the cycles (69) would be reduced to the single term w^β, which it will be convenient to regard as a cycle though it consists of only one term. In this case $k = 1$. It may be assumed that λ is less than s, h less than t, and so on as regards all the numbers s, t, etc., in (67) which are odd primes. The numbers λ, h, etc., are prime roots of s, t, etc., respectively. Take P_1 a rational function of w, and, z being any integer, let P_z be what P_1 becomes when z is changed into w^z. Put

$$\left.\begin{array}{l} \phi_\sigma = P_\sigma^{\lambda^{s-2}} P_{\sigma\lambda}^{\lambda^{s-3}} P_{\sigma\lambda^2}^{\lambda^{s-4}} \dots\dots P_{\sigma\lambda^{s-2}} \\ \psi_\tau = P_\tau^{h^{t-2}} P_{\tau h}^{h^{t-3}} P_{\tau h^2}^{h^{t-4}} \dots\dots P_{\tau h^{t-2}} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ X_\delta = P_\delta^{l^{d-2}} P_{\delta l}^{l^{d-2}} P_{\delta l^2}^{\lambda^{d-4}} \dots\dots P_{\delta l^{d-2}} \\ F_\beta = P_\beta^{k^{b-3}} P_{\beta k}^{k^{b-3}} P_{\beta k^2}^{k^{b-4}} \dots\dots P_{\beta k^{b-2}} \end{array}\right\} \tag{70}$$

In the case when one of the numbers in (67), say b, is 2, the last of equations (70) is reduced to

$$F_\beta = P_\beta. \tag{71}$$

Then, if R_1 be the fundamental element of the root of a pure uni-serial Abelian equation $f(x) = 0$ of the n^{th} degree, it will be found that

$$R_1 = A_1^n\left(\phi_\sigma^\sigma \psi_\tau^\tau \dots\dots X_\delta^\delta F_\beta^\beta\right), \tag{72}$$

A_1 being a rational function of w.