proved the proposed cases to belong to the irreducible cubic, and I add their roots calculated to twenty places of decimals.

Yours respectfully,

J. C. GLASHAN.

BOTHWELL, May 31st, 1870.

I. $x^2+y=a \therefore y^2 = x^4 - 2ax^2 + a^2$

$x+y^2=b \therefore y^2 = b - x$

$\therefore x^4 - 2ax^2 + x + (a^2 - b) = 0.$

The discriminant of this quadratic is

$$\frac{(4a^2-3b)^3}{3} - \frac{27\,(144ab-128a^3-27)^2}{432}$$

(See *Salmon's Lesson on Higher Algebra*, second edition Art. 203.)

This, with its sign changed, is the discriminant of the canonizing cubic (Sal. Art. 169: After i in the note, insert, *with sign changed.* See also Arts. 207 and 208.) The cubic can be reduced to its canonical form (primary or secondary) by real transformation, if its discriminant is positive or vanishes. (Sal. Arts. 164 and 104: I call the form x^2y *the secondary canonical* of the binary cubic with a square factor.) Therefore, if the above discriminant is negative or vanishes the reducing cubic, and in consequence the proposed equations can be solved, but if it is positive, the equations cannot be solved by any *direct and inartificial* means. In the former case x and y will have each two real and two imaginary values, or else two *equal values*

II. The above discriminant will certainly be negative, and, therefore, the equations reducible if $3b =$ or $> 4a^2$. This, then, may be taken as a *first* test. If it fails, substitution in the discriminant must be made.

III. If there is one positive value of x greater than 1, simultaneous with one of y also greater than 1, satisfying the proposed equations, they are irreducible, and x and y have each four real values.

The above discriminant, neglecting positive numerical *multipliers*, which do not affect its sign, reduces to $256\,(a^2-b)\,(b^2-a) + (32ab-27)$. Under the condition proposed, the latter term and each factor of the former term is positive, therefore the discriminant is positive, and as x and y have each one real value they have four. (Sal. Art. 209.)

The above propositions are true not only for the equation in x derived from the proposed equations, but also for any and every equation in x or y or both, derived from them. The *form* of the proof secures this. (Or see Sal. Art. 202.) Since the discriminant is an invarient, a and b may be interchanged in II.

EXAMPLES.

$x^2+y=11$, $x+y^2=7$ — $x=3$ and $y=2$ satisfy the equations, therefore they are irreducible.

The values of x and y to twenty decimals are—

$x = 3$,
or $3{\cdot}58442834033049174494$,
or $-3{\cdot}77931025337774689189$,
or $-2{\cdot}80511808695274485305$,
$y = 2$,
or $-1{\cdot}84812652696440355354$,
or $-3{\cdot}28318599128616941226$,
or $3{\cdot}13131251825057296580$.

$x^2+y=\frac{16}{9}$, $x+y^2=\frac{22}{9}$ — The discriminant is positive and the equations are irreducible.

The roots are $x=\frac{2}{3}$ or $-\frac{1}{3}$ or $\dfrac{-1 \pm 3\sqrt{13}}{6}$

$y = \frac{4}{3}$ or $\frac{5}{3}$ or $\dfrac{-9 \pm \sqrt{13}}{6}$

$x^2+y=\frac{7}{9}$, $x+y^2=\frac{7}{3}$ — $3b > 4a^2 \therefore$ the equations are reducible.

The roots are $x=\frac{4}{3}$ or $-\frac{5}{3}$ or $\dfrac{1 \pm 3\sqrt{-3}}{6}$

$y = -1$ or 2 or $\dfrac{9 \pm \sqrt{-3}}{6}$

$x^2+y=1\frac{1}{16}$, $x+y^2=1\frac{1}{4}$ — The discriminant vanishes, and $\therefore$ two of the roots are equal.

The roots are $x=\frac{1}{4}, \frac{1}{4}$, or $\dfrac{-1 \pm 4\sqrt{2}}{4}$

$y = 1, 1,$ or $\dfrac{-2 \pm \sqrt{2}}{2}$

The following is the calculation of the positive incommensurable root of x in the first example:—

1 + 3	−13	−38[3·58442834033049174494
+ 3	+18	+15
6	+ 5	−23000
3	+27	19125
9	3200	−3875000
3	625	3666912
125	3825	−208088000
5	650	187936704
130	447500	−20151296000
5	10864	18817867584
1358	458364	−1333428416000
8	10928	941008902888
1366	46929200	−392419513112000
8	54976	376406641882752
13744	46984176	−16012871229248000
4	54992	14115283316261187
13748	4703916800	−1897587912986813
4	550096	1882037962546163
137524	4704466896	−15549950440650
4	550112	14115284885373
137528	470501700800	−1434665555277
4	2750644	1411528488673
1375322	470504451444	−23137066604
2	2750648	18820379849
1375324	47050720209200	−4316686755
2	110026144	4234585466
13753268	47050830235344	−82101289
8	110026208	47050950
13753276	4705094026155200	−35050339
8	412598529	32935665
137532843	4705094438753729	−2114674
3	412598538	1882038
137532846	470509485135226,7	−232636
3	5501314	188204
137532849	470509490636540,7	−44432
	5501314	42346
	4705094961378,54	−2086
	413	1882
	4705094961791	−204
	413	188
	470509496220,4	−16
	4	
	470509496224,4	
	4	
	470509496228	

I propose these,—

I. $x+y=-1$, $x^6+y^6=-2$

II. $x+y=1$, $x^8+y^8=2\frac{1}{3}$

III. $x+y=\dfrac{1}{xy}$, $x-y=xy$

IV. $4xy^2=5\,(5-x)$, $2\,(x^2+y^2)=5$

V. $(x-5)^5+(3-y)^5+32=0$

VI. $2x^3=(x-6)^2$

III. Papers on Figures and Statistics.

1. COUNTING FOUR HUNDRED MILLIONS.

A writer thus undertakes to convey some idea of the greatness of the population of China: "The mind cannot grasp the import of so vast a number. Four hundred millions! What does it mean. Count it. Night and day, without rest, or food, or sleep, you continue the wearisome work; yet eleven days have passed before you count the first million, and more than as many years before the end of the tedious task can be reached." He also supposes this mighty multitude to take up its line of march, in a grand procession, placed in a single file at eight feet apart, and marching at the rate of thirty miles per day, except on Sabbath, which is given to rest. "Day after day the moving columns advances; the head pushing on toward the rising sun, now bridges the Pacific, now bridges the Atlantic. And now the Pacific is recrossed, but still the long proces-