$$F(m) = 1 + \frac{m}{1!} + \frac{m^2}{2!} + \ldots\ldots + \frac{m^r}{r!} + \ldots\ldots$$

$$F(n) = 1 + \frac{n}{1!} + \frac{n^2}{2!} + \ldots\ldots + \frac{n^r}{r!} + \ldots\ldots$$

Assuming that these series may be multiplied as if they were two polynomials we have

$$F(m) . F(n) = 1 + \frac{1}{1!}(m+n) + \text{terms of higher dimensions in } m \text{ and } n.$$

In this product the term of r dimensions in m and n is

$$\frac{m^r}{r!} + \frac{m^{r-1}}{(r-1)!} \cdot \frac{n}{1!} + \frac{m^{r-2}}{(r-2)!} \cdot \frac{n^2}{2!} + \ldots\ldots + \frac{n^r}{r!}$$

which can be put in the form

$$\frac{1}{r!}\left[m^r + \frac{r}{1} m^{r-1}n + \frac{r(r-1)}{1.2} m^{r-2}n^2 + \ldots\ldots + n^r\right]$$

which is equal to $\dfrac{(m+n)^r}{r!}$.

Then giving to r the values 1, 2, 3, ... we have

$$F(m) . F(n) = 1 + \frac{(m+n)}{1!} + \frac{(m+n)^2}{2!} + \ldots\ldots + \frac{(m+n)^r}{r!} + \ldots\ldots$$

that is, the product is the result of putting $m+n$ in place of (x) in (I). Therefore,

$$F(m) . F(n) = F(m+n). \qquad (II)$$

Let, now, x be a positive integer. Then, by repeated application of (II), it follows that

$$F(1) . F(1) \ldots\ldots \text{to } x \text{ factors} = F(1 + 1 + \ldots\ldots \text{to } x \text{ terms}).$$

$$\therefore \left\{F(1)\right\}^x = F(x) \qquad (III)$$

Next let $\frac{p}{q}$ be any positive fraction, p and q being integers. Then

$$F\left(\frac{p}{q}\right) . F\left(\frac{p}{q}\right) \ldots\ldots \text{to } q \text{ factors} = F\left(\frac{p}{q} + \frac{p}{q} + \ldots\ldots \text{to } q \text{ terms}\right)$$

$$\therefore \left\{F\left(\frac{p}{q}\right)\right\}^q = F(p) = \left\{F(1)\right\}^p, \text{ by } (III) \text{ since } p \text{ is a positive integer.}$$