(3) subsists for every integral value of z, an equation (5), in which the symbol F has the same meaning as in (3), at the same time subsisting for every value of e prime to n, the n values of x_{s+1} in (2), obtained by giving s successively the values $0, 1, 2, \ldots, n-1$, are the roots of a pure uni-serial Abelian equation, provided always that the equation of the n^{th} degree, of which they can be shown to be the roots, is irreducible.

Proof of the Criterion.

§ 11. Here we assume that the conditions specified in § 10 are satisfied, and we have to show that the n values of x_{s+1} in (2), obtained by putting s successively equal to $0, 1, 2, \ldots, n-1$, are the roots of a pure uni-serial Abelian equation.

§ 12. We will first prove that the n values of the expression (4) obtained by giving t successively the n values $0, 1, 2, \ldots, n-1$, are the same, the order of the terms not being considered, as the n values of x_{s+1} in (2) obtained by giving s successively the values $0, 1, 2, \ldots, n-1$.

Because w^e is a primitive n^{th} root of unity, all the n^{th} roots of unity distinct from unity are contained in the series

$$w^e,\ w^{2e},\ w^{3e},\ \ldots,\ w^{(n-1)e}.$$

Therefore the two series

$$R_1,\ R_2,\ R_3,\ \ldots,\ R_{n-1},$$
$$R_e,\ R_{2e},\ R_{3e},\ \ldots,\ R_{(n-1)e},$$

are identical with one another, the order of the terms not being considered. Therefore, also, the two series

$$R_1^{\frac{1}{n}},\ R_2^{\frac{1}{n}},\ R_3^{\frac{1}{n}},\ \ldots,\ R_{n-1}^{\frac{1}{n}},$$
$$R_e^{\frac{1}{n}},\ R_{2e}^{\frac{1}{n}},\ R_{3e}^{\frac{1}{n}},\ \ldots,\ R_{(n-1)e}^{\frac{1}{n}},$$

are identical with one another, the order of the terms not being considered, it being understood that $R_e^{\frac{1}{n}}$, $R_{2e}^{\frac{1}{n}}$, etc., are the same n^{th} roots of R_e, R_{2e}, etc., or of R_1, R_2, etc., that are taken in the series $R_1^{\frac{1}{n}}$, $R_2^{\frac{1}{n}}$, etc. Let the expression (4) be called x'_{t+1}. The separate members of the expression x_{s+1} are

$$R_0^{\frac{1}{n}},\ w^s R_1^{\frac{1}{n}},\ w^{2s} R_2^{\frac{1}{n}},\ \text{etc.} \qquad (13)$$

Taking s with a definite value, let

$$es = bn + c,$$