$$\left.\begin{aligned} &\text{Therefore} \quad (R_z R_1^{-z})^{\frac{1}{n}} = (A_z A_1^{-z})(\phi_{z\sigma}\phi_\sigma^{-z})^{\frac{\sigma}{n}} \; \ldots . \; (F_{z\beta}F_\beta^{-z})^{\frac{\beta}{n}} \\ &\text{and} \quad (R_{ez} R_e^{-z})^{\frac{1}{n}} = (A_{ez} A_e^{-z})(\phi_{ez\sigma}\phi_{e\sigma}^{-z})^{\frac{\sigma}{n}} \; \ldots . \; (F_{ez\beta}F_{e\beta}^{-z})^{\frac{\beta}{n}} \end{aligned}\right\} \quad (97)$$

Because $(\phi_{z\sigma}\phi_\sigma^{-z})^{\frac{\sigma}{n}}$ and other corresponding expressions have been shown to be rational functions of the primitive n^{th} root of unity w, the two equations (97) correspond respectively to (3) and (5). If z be not prime to n, and yet not a multiple of n, it may be taken to be ev, where v is equal to $\frac{n}{y}$, y being one of the terms in the series (75) distinct from n, and w^e being the general primitive n^{th} root of unity. Then, just as we obtained the pair of equations (97) by means of (76), we can now, by means of (77), obtain the pair of equations

$$\left.\begin{aligned} (R_{ev} R_1^{-ev})^{\frac{1}{n}} &= (A_{ev} A_1^{-ev})(\phi_{ev\sigma}\phi_\sigma^{-ev})^{\frac{\sigma}{n}} \ldots . \\ (R_{cev} R_c^{-ev})^{\frac{1}{n}} &= (A_{cev} A_c^{-ev})(\phi_{cev\sigma}\phi_{c\sigma}^{-ev})^{\frac{\sigma}{n}} \ldots . \end{aligned}\right\} \quad (98)$$

where w^c represents any one of the primitive n^{th} roots of unity. Because such expressions as $(\phi_{ev\sigma}\phi_\sigma^{-ev})^{\frac{\sigma}{n}}$ and $(\phi_{cev\sigma}\phi_{c\sigma}^{-ev})^{\frac{\sigma}{n}}$ are rational functions of w, the two equations (98) correspond respectively to (3) and (5). Finally, should z be a multiple of n, it may be taken to be zero. Then the equation corresponding to (3) is, q_1 being a rational function of w,

$$R_z^{\frac{1}{n}} = q_1 R_1^{\frac{z}{n}}; \text{ or, since } z = 0,\ R_0^{\frac{1}{n}} = q_1.$$

But $R_0^{\frac{1}{n}}$ is rational. Therefore q_1 is rational. Therefore $q_1 = q_e$; in other words, q_1 undergoes no change when w becomes w^e. Also $R_{ez}^{\frac{1}{n}} = R_0^{\frac{1}{n}} = q_e$. Therefore, since $R_e^{\frac{z}{n}} = 1$,

$$R_{ez}^{\frac{1}{n}} = q_e R_e^{\frac{z}{n}},$$

which is the equation corresponding to (5). Therefore, whatever z be, the equation (5) subsists along with (3). Hence, by the Criterion in § 10, the expression (73) is the root of a pure uni-serial Abelian equation of the n^{th} degree.

The Pure Uni-Serial Abelian of a Degree which is Four Times the Continued Product of a Number of Distinct Odd Primes.

Fundamental Element of the Root.

§ 47. Let $n = 4m$, where m is the continued product of the distinct odd prime numbers,

$$s,\ t,\ \ldots .,\ d,\ b. \quad (99)$$

Take

$$\sigma,\ \tau,\ \ldots .,\ \delta,\ \beta, \quad (100)$$