Solving for the coordinates of $\mathbf{Q}_3$, and collecting terms,

$$\mathbf{Q}_3 = (b^2 S_{ac} - S_{ab} S_{bc}) \frac{\mathbf{A}}{v^2} + (a^2 S_{bc} - S_{ab} S_{ac}) \frac{\mathbf{B}}{v^2},$$

$$q_3^2 = q_x^2 + q_y^2 + q_z^2 + q_u^2$$

$$= (a^2 S_{bc}^2 + b^2 S_{ac}^2 - 2 S_{ab} S_{ac} S_{bc}) \frac{1}{v^2},$$

where $\quad v^2 = a^2 b^2 - S_{ab}^2$.

Then

$$\mathbf{N}_3 = \mathbf{C} - \mathbf{Q}_3 = \begin{vmatrix} S_{aa} & S_{ab} & S_{ac} \\ S_{ba} & S_{bb} & S_{bc} \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{vmatrix} \div \begin{vmatrix} S_{aa} & S_{ab} \\ S_{ba} & S_{bb} \end{vmatrix}$$

$$n_3^2 = c^2 - q_3^2 = \begin{vmatrix} S_{aa} & S_{ab} & S_{ac} \\ S_{ba} & S_{bb} & S_{bc} \\ S_{ca} & S_{cb} & S_{cc} \end{vmatrix} \div \begin{vmatrix} S_{aa} & S_{ab} \\ S_{ba} & S_{bb} \end{vmatrix}$$

$$= \frac{w^2}{v^2}.$$

These forms are identical for 3-space, and apparently for all space above it.

In 3-space also $\quad n_3 = \dfrac{| a_x \, b_y \, c_z |}{v}$.

41. To find the normal $\mathbf{N}_4$, from $\mathbf{D}$ to the 3-flat of $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

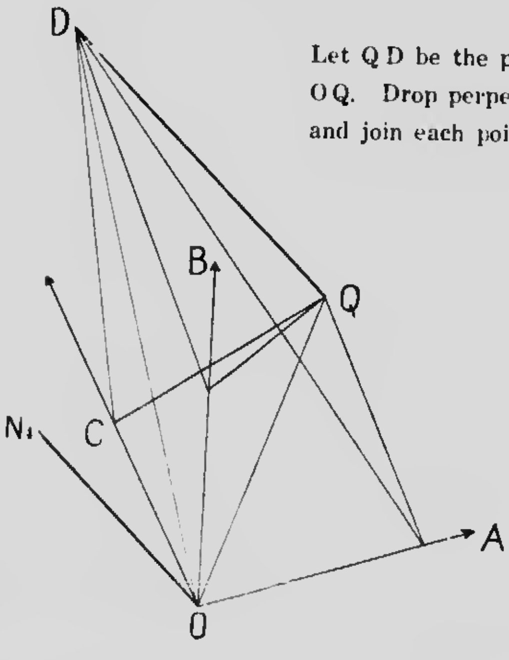

Fig. 12

Let QD be the positive direction of $\mathbf{N}_4$. Join OQ. Drop perpendiculars from $\mathbf{Q}_4$ on $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and join each point of intersection with $\mathbf{D}$.

Then it is evident as in §40 that

$$S_{aq} = S_{ad} \quad \ldots\ldots\ldots (1)$$
$$S_{bq} = S_{bd} \quad \ldots\ldots\ldots (2)$$
$$S_{cq} = S_{cd} \quad \ldots\ldots\ldots (3).$$

Since $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{Q}_4$ are all in the same 3-flat and therefore all perpendicular to $\mathbf{N}_4$

$S_{an} = S_{bn} = S_{cn} = S_{qn} = 0.$

Eliminating the n's we get the cosolid equation

$$| a_x \, b_y \, c_z \, q_u | = 0 \ldots (4).$$