A PISTON PROBLEM.

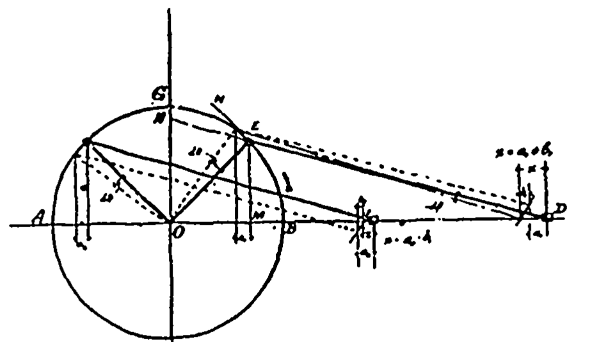
Editor CANADIAN ENGINEER :

DEAR SIR,—Will you kindly answer the following question: Does a piston of a steam engine that is connected in the usual way by crosshead and connecting rod to a crank plate, travel faster in one end of a cylinder than in the other, the fly wheel running at a regular speed? Yours truly,

A LONDON SUBSCRIBER

G. Sinclair Smith, of McGill University, answers this question as follows:—

To simplify the figure, though it does not affect the reasoning or result, suppose the piston rod to be omitted, and the piston at D simply attached to the end of the connecting rod E D.



Also suppose the crank to rotate uniformly, as stated, in direction of the arrow. Let the length of the crank be taken as unity and the connecting rod as *n* cranks in length. Suppose the forward stroke to start from B, then G is the end of the first quadrant and A that of the second.

The motion of the piston consists of two parts, one (a) due to the motion of the crank through an angle $d\theta$, and the other (b) due to the motion of the connecting rod through an angle $d\phi$. Until the connecting rod occupies its position of maximum obliquity these two quantities are additive, as shown in the accompanying figure; after that, until the position of maximum obliquity in the return stroke, they are subtractive.

The speed of the piston would be exactly the same at similar positions on each side of the vertical centre line, if the connecting rod were infinitely long [as, *e.g.*, in some steam pumps where a slotted crosshead and sliding block take the place of the connecting rod.]

The ordinary short connecting rod, however, induces, as we see, a want of symmetry in the motion; the outward swing producing an increase, the inward a diminution in the speed of the piston.

Thus the piston starts *from rest* at the beginning of the stroke, and gradually increases its speed until it attains a maximum, when the connecting rod and crank are approximately at right angles. It then gradually falls off until the piston again comes to rest, when the crank pin reaches A.

Let us now examine this analytically and find the actual velocity of the piston for any position of the crank.

Using the same figure as before with the dotted position of the crank, call the angle that the crank makes with the line of centres θ, and that which the connecting rod makes α. Suppose we let v represent the piston velocity and V the crank-pin velocity, then since the direction of motion of V at E is tangential along EN and the connecting rod ED is rigid, the resolved velocity of V along ED must equal the resolved velocity of v along ED, that is,

$$V \cos NEH = v \cos \alpha$$

$$\text{or, } V \sin HEO = v \cos \alpha$$

$$\text{or, } V \sin(\alpha + \theta) = v \cos \alpha$$

$$\therefore \frac{v}{V} = \frac{\sin(\alpha+\theta)}{\cos \alpha}$$

$$= \frac{\sin \alpha \cos \theta + \cos \alpha \sin \theta}{\cos \alpha} \qquad (1)$$

$$\text{Now since } \frac{\sin \alpha}{\sin \theta} = \frac{OE}{ED} = \frac{1}{n} \therefore \sin \alpha = \frac{\sin \theta}{n} \qquad (2)$$

$$\text{and } \cos \alpha = \sqrt{1 - \frac{\sin^2 \theta}{n^2}} = \frac{1}{n}\sqrt{n^2 - \sin^2 \theta} \qquad (3)$$

Substituting in (1) values of $\sin \alpha$ and $\cos \alpha$ from (2) and (3),

$$\text{we get } \frac{v}{V} = \frac{\frac{1}{n} \sin \theta \cos \theta + \frac{1}{n} \sin \theta \sqrt{n^2 - \sin^2 \theta}}{\frac{1}{n} \sqrt{n^2 - \sin^2 \theta}}$$

$$\text{or, } = \sin \theta \frac{[\sqrt{n^2 - \sin^2 \theta} + \cos \theta]}{\sqrt{n^2 - \sin^2 \theta}}$$

$$\text{or, } = \sin \theta + \frac{\sin^2 \theta}{2\sqrt{n^2 - \sin^2 \theta}} \qquad (4)$$

Suppose that n here = 4, which is a common ratio of connecting rod to crank, then by substituting this and the value of θ for any desired position of the crank, the speed v of the piston can be found in terms of V the speed of the crank pin. Suppose we take similar positions of the crank at both ends; 1st, at 5°—substituting this in the formula (4), we get $\frac{v}{V} = .1089$, or the piston moves at about $\frac{1}{10}$ of the speed of the crank pin at this point. Now at 5° from the other end, that is, at 175° from the beginning of the stroke, we find $\frac{v}{V} = .0655$, or the piston moves at about $\frac{1}{16}$ of the speed of the crank pin at this point. Again, taking positions of the crank at 10° from each end of the stroke, we find that at the inner end the piston moves at .2164 times the speed of the crank pin, while at the outer end it moves at .131 times the speed of the crank pin. From this it is seen that the piston moves much faster in the first part of the forward stroke than in the second part, and *vice versa* in the return stroke.

Now, since the piston moves faster in the first part of the forward stroke than in the last, it evidently gets to the middle of its stroke before the crank pin has reached the end of the first quadrant. This evidently occurs when the distance from centre O of crank shaft to crosshead is equal to the length of the connecting rod. In the figure drop a perpendicular EM from E to OD. Now, the distance OD must equal the length of the connecting rod, that is, equal *n*.

$$\text{Now } OD = OM + MD = \cos \theta + n \cos \alpha \text{ but}$$

$$\cos \alpha = \frac{1}{n}\sqrt{n^2 - \sin^2 \theta} \text{ from (3) } \therefore OD = \cos \theta + \sqrt{n^2 - \sin^2 \theta}$$

$$\text{or, } n = \cos \theta + \sqrt{n^2 - \sin^2 \theta}$$

$$\text{or, } n - \cos \theta = \sqrt{n^2 - \sin^2 \theta}$$

$$\text{or, } n^2 - 2n \cos \theta + \cos^2 \theta = n^2 - \sin^2 \theta$$

$$\therefore 2n \cos \theta = \sin^2 \theta + \cos^2 \theta$$

$$= 1$$

$$\text{and, } \cos \theta = \frac{1}{2n}$$

or, the piston is in the middle of its stroke when $\cos \theta = \frac{1}{2n}$. Taking as before n = 4, then $\cos \theta = .125$ and $\theta = 82° 50'$. Thus the first half of the forward stroke is made in $\frac{82\frac{5}{6}}{180}$ or about $\frac{46}{100}$ of the time in which the whole stroke is performed in an engine where the connecting rod is four times the length of the crank. As the value of *n* increases, the fluctuation in the