7. (*a*). Factor $a^4 + 4b^4$

$a^4 + 4b^4 = a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2$

$= (a^2 + 2b^2)^2 - (2ab)^2 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$

$= (a^2 + 2ab + b^2 - i^2b^2)(a^2 - 2ab + b^2 - i^2b^2)$

$= (a + b + ib)(a + b - ib)(a - b + ib)(a - b - ib)$; where $i = \sqrt{-1}$.

Or as follows—

$a^4 + 4b^4 = a^4 - 4i^2b^4 = (a^2 - 2ib^2)(a^2 + 2ib^2)$

$= [a^2 - (1 + i)^2b^2]\,[a^2 - (1 - i)^2b^2]$

$= [a + b(1 + i)]\,[a - b(1 + i)]\,[a + b(1 - i)]\,[a - b(1 - i)]$.

Factor $a^3 + b^3 + c^3 - 3abc$.

The factors of this are known, or should be known to every algebraist, as it constitutes an important fundamental form.

To factor it, however, we may do so as follows :—

The expression is homogeneous and symmetrical, so that if it has a linear factor it must be $a + b + c$. For a write $-(b + c)$, and the expression vanishes.

$\therefore$ $a + b + c$ is a linear factor.

The quadratic co-factor must he homogeneous and symmetrical, and must be of the form $a^2 + b^2 + c^2 + (ab + bc + ca)\,m$, and

$(a + b + c)(a^2 + b^2 + c^2 + [bc + ca + ab]\,m)$ must beidentical with $a^3 + b^3 + c^3 - 3abc$.

Comparing a term as a^2b, not found in the second expression, its coefficient from the first expression is $1 + m$ and is zero $\therefore m = -1$, and the quadratic factor is

$$a^2 + b^2 + c^2 - bc - ca - ab.$$

Factor $(1 + y)^2 - 2x^2(1 + y^2) + x^4(1 - y)^2$.

The substitutions $x = 1$ and $x = -1$ cause this to vanish, and therefore $x^2 - 1$ is a factor. By division we find the other factor to be

$$x^2(1 - y)^2 - (1 + y)^2;$$

which being the difference of two squares is readily factored.

b. If $s = \frac{1}{2}(a + b + c)$, show that

$$(s - b)(s - c) + (s - c)(s - a) + (s - a)(s - b) = s^2 - \frac{a^2 + b^2 + c^2}{2}$$

$(s - b)(s - c) = s^2 - s(b + c) + bc$.

$\therefore$ $(s - c)(s - a) = s^2 - s(c + a) + ca$, by symmetry,

and $(s - a)(s - b) = s^2 - s(a + b) + ab$.

$\therefore$ $(s - b)(s - c) + (s - c)(s - a) + (s - a)(s - b) = 3s^2 - s.4s + \Sigma ab. = \Sigma ab - s^2$.

But $4s^2 = (a + b + c)^2 = \Sigma a^2 + 2\,\Sigma ab$.

$\therefore$ $\Sigma ab = 2s^2 - \frac{1}{2} 2a^2$.

Whence, $\Sigma ab - s^2 = s^2 - \frac{1}{2}\Sigma a^2$, which establishes the required relation.

8. *a*. Find the square root of $x^4 + 4x^3y + 10x^2y^2 + 12xy^3 + 9y^4$.

(Note. By a typographical error the fourth term was printed " $12xy$ ").

x^2 and $3y^2$ are two terms of the root.

Assume $x^2 + axy + 3y^2$ to be the root. The term containing x^3y is $2ax^3y$.

$\therefore$ $2a = 4$ and $a = 2$, and the root is $\pm(x^2 + 2xy + 3y^2)$.

(*b*). If $x^2 + px + g$, and $x^2 + mx + n$ have a common factor, to find the relation that exists among p, g, m, n.

There are many ways of doing this, the following is about the simplest :