The Root Constructed from its Fundamental Element.

§41. From R_1, as expressed in (72), derive R_0, R_2, etc., by changing w into w^0, w^2, etc. By §5, the root of the equation $f(x) = 0$ is

$$R_0^{\frac{1}{n}} + R_1^{\frac{1}{n}} + R_2^{\frac{1}{n}} + \ldots . + R_{n-1}^{\frac{1}{n}}. \qquad (73)$$

To construct the root, we have to determine the particular n^{th} roots of R_0, R_1, etc., that are to be taken together in (73). When w is changed into w^z, let A_1, ϕ_1, ψ_1, etc., become A_z, ϕ_z, ψ_z, etc., respectively. Then

$$\left.\begin{aligned} R_z &= A_z^n(\phi_{z\sigma}^{\sigma}\psi_{z\tau}^{\tau} \ \ldots . X_{z\delta}^{\delta}F_{z\beta}^{\beta}) \\ \text{therefore} \qquad R_z^{\frac{1}{n}} &= w'A_z(\phi_{z\sigma}^{\sigma}\psi_{z\tau}^{\tau} \ldots . X_{z\delta}^{\delta}F_{z\beta}^{\beta})^{\frac{1}{n}} \end{aligned}\right\} \qquad (74)$$

w' being an n^{th} root of unity. Let the integers not greater than n that measure n, unity not included, be

$$n,\ y,\ \text{etc.} \qquad (75)$$

For instance, if $n = 3 \times 5 \times 7 = 105$, the series (75) is

$$105,\ 35,\ 21,\ 15,\ 7,\ 5,\ 3.$$

The n^{th} roots of unity distinct from unity are the primitive n^{th} roots of unity, the primitive y^{th} roots of unity, and so on. For instance, the series of the 105^{th} roots of unity distinct from unity, containing 104 terms, is made up of the 48 primitive 105^{th} roots of unity, the 24 primitive 35^{th} roots of unity, the 12 primitive 21^{st} roots of unity, the 8 primitive 15^{th} roots of unity, the 6 primitive 7^{th} roots of unity, the 4 primitive 5^{th} roots of unity, and the 2 primitive 3^{d} roots of unity. The general primitive n^{th} root of unity being w^e, give w' in the second of equations (74) the value unity for every value of z included under e. Then

$$R_e^{\frac{1}{n}} = A_e(\phi_{e\sigma}^{\sigma}\psi_{e\tau}^{\tau} \ldots . X_{e\delta}^{\delta}F_{e\beta}^{\beta})^{\frac{1}{n}}. \qquad (76)$$

Taking any other term than n, say y, in the series (75), since y is a factor of n, let $yv = n$. Then w^v is a primitive y^{th} root of unity. Hence, since w^e is the general primitive n^{th} root of unity, all the primitive y^{th} roots of unity are included in w^{ev}. If w', in the second of equations (74), be w^a when $z = v$, let it have the value w^{ea} when $z = ev$. Then

$$R_{ev}^{\frac{1}{n}} = w^{ea}A_{ev}(\phi_{ev\sigma}^{\sigma}\psi_{ev\tau}^{\tau} \ldots . X_{ev\delta}^{\delta}F_{ev\beta}^{\beta})^{\frac{1}{n}}. \qquad (77)$$

Form equations similar to (77) for the remaining terms in (75). In this way, because the series of the n^{th} roots of unity distinct from unity is made up of the primitive n^{th} roots of unity, the primitive y^{th} roots of unity, and so forth, all the terms $1, 2, \ldots . , n-1$ are found in the groups of numbers represented