# UNIVERSITY WORK.

## MATHEMATICS.

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### PROBLEM SOLUTION.

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Prove that

$$\frac{a^{n+2}(b-c)+b^{n+2}(c-a)+c^{n+2}(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}=$$

the sum of all terms of $n$ dimensions that can be made by the letters $a$, $b$, $c$. This problem occurs in Wolstenholme, and is of considerable practical importance in obtaining the factors of a certain class of quantities. The following solution suggests itself to me:

(1) $\frac{1}{1-ax}=1+ax+a^2x^2+a^3x^3+\ldots\ldots$

(2) $\frac{1}{1-bx}=1+bx+b^2x^2+b^3x^3+\ldots\ldots$

(3) $\frac{1}{1-cx}=1+cx+c^2x^2+c^3x^3+\ldots\ldots$

$$\therefore \frac{1}{(1-ax)(1-bx)(1-cx)}=$$

$$(1+ax+a^2x^2+\ldots)(1+bx+b^2x^2+\ldots)(1+cx+c^2x^2).$$

From the right-hand product select the co-efficient of $x^n$, the result will be $a^n+b^n+c^n+a^{n-1}b+a^{n-1}c+\ldots\ldots$ or the sum of all the terms of $n$ dimensions in $a$, $b$ and $c$.

Let $\frac{1}{(1-ax)(1-bx)(1-cx)}=\frac{A}{1-ax}+\frac{B}{1-bx}+\frac{C}{1-cx}$, clear of fractions and equate co-efficients. It will be found that

$$A=\frac{n^2}{(a-b)(a-c)};\ B=\frac{b^2}{(b-c)(b-a)};$$

$$C=\frac{c^2}{(c-a)(c-b)};$$

$$\therefore \frac{1}{(1-ax)(1-bx)(1-cx)}=$$

$$\frac{a^2}{(a-b)(a-c)}(1-ax)^{-1}+\frac{b^2}{(b-c)(b-a)}(1-bx)^{-1}+\frac{c^2}{(c-a)(c-b)}(1-cx)^{-1}.$$

Expand $(1-ax)^{-1}$, $(1-bx)^{-1}$, $(1-cx)^{-1}$, and select from each expansion the co-efficient of $x^n$. The co-efficient of $x^n$ in $(1-ax)^{-1}$ is $a^n$; the co-efficient of $x^n$ in $(1-bx)^{-1}$ is $b^n$, etc.

$\therefore$ the co-efficient of $x^n$ in

$$\frac{1}{(1-ax)(1-bx)(1-cx)} \text{ is } \frac{a^{n+2}}{(a-b)(a-c)}+\frac{b^{n+2}}{(b-c)(b-a)}+\frac{c^{n+2}}{(c-a)(c-b)} \text{ or}$$

$$\frac{a^{n+2}(b-c)+b^{n+2}(c-a)+c^{n+2}(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$$

(since $(a-b)(b-c)(a-c)=a^2(b-c)+b^2(c-a)+c^2(a-b)$).

Therefore

$$\frac{a^{n+2}(b-c)+b^{n+2}(c-a)+c^{n+2}(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$$

is equal to sum of all terms of $n$ dimensions in $a$, $b$ and $c$.—Q.E.D.

The practical use of this theorem may be shown thus: Suppose we are asked to find the factors of $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

Put $n=1$ then $\frac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$
$=a+b+c$;

$\therefore a^3(b-c)+b^3(c-a)+c^3(a-b)$
$=(a-b)(b-c)(a-c)(a+b+c)$.

Similarly $a^4(b-c)+b^4(c-a)+c^4(a-b)=(a-b)(b-c)(a-c)(a^2+b^2+c^2+ab+bc+ca)$; $a^5(b-c)+b^5(c-a)+c^5(a-b)=(a-b)(b-c)(a-c)\left\{a^3+b^3+c^3+a^2b+ab^2+a^2c+ac^2+b^2c+bc^2+abc\right\}$; $a^6(b-c)+$ and $+$ and $=(a-b)(b-c)(a-c)\left\{a^4+b^4+c^4+a^3b+ab^3+\ldots\right\}$.