§5. If an algebrical expression r_1, arranged as in (1), be zero, while the coefficients g_1, k_1, etc., are not all zero, an equation

$$\omega \Delta_1^{\frac{1}{m}} = l_1 \tag{2}$$

must subsist; where ω is an m^{th} root of unity; and l_1 is an expression involving only such surds exclusive of $\Delta_1^{\frac{1}{m}}$ as occur in r_1. For, let the first of the coefficients h_1, e_1, etc., proceeding in the order of the descending powers of $\Delta_1^{\frac{1}{m}}$, that is not zero, be n_1, the coefficient of $\Delta_1^{\frac{s}{m}}$. Then we may put

$$mr_1 = n_1 \left\{ f\left(\Delta_1^{\frac{1}{m}} \right) \right\} = n_1 \Delta_1^{\frac{s}{m}} + \text{etc.} = 0.$$

Because $\Delta_1^{\frac{1}{m}}$ is a root of each of the equations $f(x) = 0$ and $x^m - \Delta_1 = 0$, $f(x)$ and $x^m - \Delta_1$ have a common measure. Let their H. C. M., involving only such surds as occur in $f(x)$ and $x^m - \Delta_1$, be $\varphi(x)$. Then, because $\varphi(x)$ is a measure of $x^m - \Delta_1$, the roots of the equation

$$\varphi(x) = x^c + p_1 x^{c-1} + p_2 x^{c-2} + \text{etc.} = 0$$

are $\Delta_1^{\frac{1}{m}}$, $\omega_1 \Delta_1^{\frac{1}{m}}$, $\omega_2 \Delta_1^{\frac{1}{m}}$, ..., $\omega_{c-1} \Delta_1^{\frac{1}{m}}$; where ω_1, ω_2, etc., are distinct primitive m^{th} roots of unity. Therefore,

$$\Delta_1^{\frac{c}{m}} (\omega_1 \, \omega_2 \ldots) (-1)^c = p_c.$$

Now c is a whole number less than m but not zero; and, by §1, m is prime. Therefore there are whole numbers n and h such that

$$\Delta_1^{\frac{cn}{m}} (\omega_1 \, \omega_2 \ldots)^n (-1)^{cn} = \Delta_1^{\frac{1}{m}} \Delta_1^{h} (\omega_1 \, \omega_2 \ldots)^n (-1)^{cn} = p_c^{n}.$$

Therefore, if $(\omega_1 \, \omega_2 \ldots)^n = \omega$, and $l_1 \Delta_1^{h} (-1)^{cn} = p_c^{n}$, $\omega \Delta_1^{\frac{1}{m}} = l_1$.

§6. Let r_1 be an algebraical expression in which no root of unity having a rational value occurs in the surd form $1^{\frac{1}{m}}$. Also let there be in r_1 no surd $\Delta_1^{\frac{1}{m}}$ not a root of unity, such that