5. Prove that

(a.) $\dfrac{2\left\{x+2+\sqrt{(x^2-4)}\right\}}{x+2-\sqrt{(x^2-4)}}=x+\sqrt{(x^2-4)}$.

(b.) $(b+c-a)a^{\frac{1}{3}}+(c+a-b)b^{\frac{1}{3}}+(a+b-c)c^{\frac{1}{3}}=(a+b+c)(a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}})-2(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}})$.

(a) By rationalizing the denominator the required result is obtained.

(b) Remove the brackets, add and subtract $a^{\frac{4}{3}}$, $b^{\frac{4}{3}}$, $c^{\frac{4}{3}}$, and the answer can easily be obtained.

6. Solve the equations—

(a.) $(b-c)(x-a)^2+(c-a)(x-b)^2+(a-b)(x-c)^2=0$.

(b.) $x+y=4xy$; $y+z=2yz$; $z+x=3zx$.

(c.) $x+y+z=0$.
$ax+by+cz=0$.
$bcx+cay+abz+(a-b)(b-c)(c-a)=0$.

(d.) $\dfrac{x-1}{x+3}+\dfrac{x-3}{x+1}+2=0$.

(a) By inspection it can be seen that $a$ is a root, and therefore $b$ and $c$.

(b) Divide each side of the given equations by $xy$, $yz$, $zx$ respectively, and we have

(1) $\dfrac{1}{x}+\dfrac{1}{y}=4$; (2) $\dfrac{1}{y}+\dfrac{1}{z}=2$; (3) $\dfrac{1}{x}+\dfrac{1}{z}=3$.

(1) − (2), $\dfrac{1}{x}-\dfrac{1}{z}=2$; add this equation to (3) and we get $x=\frac{2}{5}$; values of $y$ and $z$ are $\frac{2}{3}$ and 2 respectively.

(c) $x=b-c$, $y=c-a$, $z=a-b$.

This question can be solved by any of the ordinary methods of elimination, or by that of indeterminate multipliers, for which see Todhunter's larger Algebra, p. 120.

(d) $\dfrac{x-1}{x+3}+\dfrac{x-3}{x+1}+2=0=1-\dfrac{4}{x+3}+1-\dfrac{4}{x+1}+2$; or, $4=\dfrac{4}{x+3}+\dfrac{4}{x+1}$,

*i.e.*, $1=\dfrac{1}{x+3}+\dfrac{1}{x+1}$; $x=1\pm\sqrt{2}$.

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ALGEBRA (FIRST CLASS).

1. If in $ax^2+2bxy+cy^2$, $ku+lv$ be substituted for $x$ and $mu+nv$ for $y$, the result takes the form $Au^2+2Buv+Cv^2$. Find the value of $(B^2-AC)\div(b^2-ac)$ in terms of $k$, $l$, $m$, $n$.

On substituting as indicated in the question, we find the values of $A$, $B$ and $C$ to be respectively

$ak^2+2bkm+cm^2$, $akl+cmn+b(ku+lm)$, $al^2+2blu+cu^2$;

$\therefore B_2-AC=(ku-lm)^2(b_2-ac)$,

whence $\dfrac{B_2-AC}{b^2-ac}=(ku-lm)_2$.

2. Resolve $a(b-c)^3+b(c-a)_3+c(a-b)^3$ into factors.

Prove that

$$\frac{Au^2+Bv^2+Cw^2}{uvw}=\frac{Ax^2+By^2+Cz^2}{xyz}$$

if $u=x(By^2-Cz^2)$, $v=y(Cz^2-Ax^2)$, $w=z(Ax^2-By^2)$.

By inspection $(b-c)$ is a factor, and so by symmetry are $(c-a)$ and $(a-b)$; so also is $a+b+c$. From consideration of dimensions there can be no other literal factor. Put the expression $=m(b-c)(c-a)(a-b)(a+b+c)$, assign any numerical values to $a$, $b$ and $c$, and we find $m=1$.

Second part follows obviously from this resolution by substituting on left-hand side of equation values of $u$, $v$, $w$.

3. Extract the square root of $(a-b)^2(b-c)^2+(b-c)^2(c-a)^2+(c-a)^2(b-c)^2$, and the cube root of $4\left\{(a-b)^6+(b-c)^6+(c-a)^6-3(a-b)^2(b-c)^2(c-a)^2\right\}$.

Let $A=a-b$, $B=b-c$, then $A+B=a-c$, $\therefore$ square root of $A^2B^2+B^2(A+B)^2+A^2(A+B)^2$ is required, *i.e.*, of $(A^2+B^2)^2+2AB(A^2+B^2)+A^2B^2$, which is $A^2+B^2+AB$ or $a^2+b^2+c^2-bc-ca-ab$.

4. Eliminate $x$, $y$, $z$ from

$$ax+by+cz=1 \qquad \frac{a}{x}=\frac{b}{y}=\frac{c}{z}$$

$$k(x^2+y^2+z^2)+2(lx+my+nz)+h=0.$$

$$\frac{a}{x}=\frac{b}{y}=\frac{c}{z}=\frac{a^2+b^2+c^2}{ax+by+cz}=a^2+b^2+c^2.$$

Subst. values of $x$, $y$ and $z$ in 3rd equation and we have

$$h(a^2+b^2+c^2)+2(al+bm+cn)+k=0.$$

5. Simplify $\dfrac{a\sqrt{b}+b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$,

$$\{\sqrt{(4+3j)}+\sqrt{(4-3j)}\}^2,$$

and $\left(\dfrac{-1+j\sqrt{3}}{2}\right)^2+\dfrac{-1+j\sqrt{3}}{2}+1$

in which $j=\sqrt{(-1)}$.