Mathematical Department.

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PROBLEMS INVOLVING FRICTION.

It must be remembered that the laws of friction usually given, viz. :—

1. The friction varies as the normal pressure when the materials of the surfaces in contact remain the same.

2. The friction is independent of the extent of the surfaces in contact so long as the normal pressure remains the same.

relate to *limiting* friction, *i.e.*, motion is supposed just about to take place, and friction acts in a direction contrary to this motion. The effect of the introduction of friction into mechanical problems is to introduce an additional unknown quantity, but the above laws furnish us with an additional equation. Thus if R be the normal reaction between two rough surfaces in contact, F the friction, and c the co-efficient of friction, the additional unknown quantity is F, and the additional equation is $F = cR$, c being a known quantity determined by experiment. Beginners occasionally make mistakes in reference to what R is in this equation. Thus, if a weight (W) be supported on a rough plane of inclination a, by a force (P) inclined at an angle θ to the plane, the weight resolved perpendicular to the plane is $W \cos a$, but it must not be supposed that the friction is $cW \cos a$, for the normal reaction of the plane is not $W \cos a$. Part of the force $W \cos a$ is counterbalanced by P resolved perpendicular to the plane, *i.e.*, by $P \sin \theta$, so that the normal reaction of the plane is $W \cos a - P \sin \theta$, and the friction is this multiplied by c. In solving problems in which rough surfaces are concerned, we represent the forces acting on the body, as usual introducing the friction (F) which always acts in a direction contrary to that in which motion is supposed to take place; and then form the usual equations by resolving in perpendicular directions and taking moments, being careful not to omit the equation $F = cR$, which experiment furnishes.

1. Find the co-efficient of friction if a weight just rest on a rough plane inclined to the horizon at an angle of 60°.

Let R be the normal reaction of the plane, F the friction acting up the plane, W the weight of the body, then resolving along and perpendicular to the plane, $F = W \sin 60°$, $R = W \cos 60°$; also $F = cR$. Hence $\frac{cR}{R} = \frac{W \sin 60°}{W \cos 60°}$, or $c = \tan 60° = \sqrt{3}$.

2. A weight of 20 lbs. just rests on a rough plane inclined at an angle of 45° to the horizon; find the pressure at right angles to the plane, and the force of friction exerted.

Resolving along and perpendicular to the plane $F = 20 \sin 45°$, $R = 20 \cos 45°$; or $F = 10\sqrt{2} = R$. Here, since $F = cR$, evidently $c = 1$.

3. A weight of 10 lbs. is just supported on a rough plane whose inclination is 60° by a power of 5 lbs. acting parallel to the plane. Find the inclination of the plane on which the weight would just rest of itself.

Resolving along and perpendicular to the plane, we have $F + 5 = 10 \sin 60$, $R = 10 \cos 60°$; also $F = cR = c \times 10 \cos 60°$. Hence $10c \cos 60° + 5 = 10 \sin 60°$; $\therefore 5c + 5 = 5\sqrt{3}$, or $c = \sqrt{3} - 1$. Again, if a be the inclination of the plane when the body just rests on it supported by friction alone, $cR = 10 \sin a$, $R = 10 \cos a$; $\therefore c = \tan a$, or $a = \tan^{-1}(\sqrt{3} - 1)$.

4. A beam rests with one end on the ground, and the other in contact with a vertical wall. *Having given the co-efficient of friction for the wall and the ground, and the distances of the centre of gravity of the beam from the ends, determine the limiting inclination of the beam to the horizon.*

Let a, b, be the distances of the centre of gravity of the beam from its lower and upper ends respectively; R, S the normal reactions of the ground and wall; c, c' the co-efficient of friction for the ground and wall respectively; W the weight of the beam, and a its inclination to the horizon. At the lower end the friction (cR) acts horizontally towards the wall; at the upper end the friction ($c'S$) acts vertically upwards along the wall, the directions of friction in both cases being contrary to the direction in which motion is about to take place.

Equating the vertical and horizontal forces, we have $R + c'S = W$, $cR = S$; hence $\frac{S}{c} + c'S = W$, or $S = \frac{cW}{1 + cc'}$. Also taking motion about the lower end, $Wa \cos a = (a + b)(S \sin a + c'S \cos a)$, or $Wa \cos a = (a + b)(\sin a + c' \cos a)\frac{cW}{1 + cc'}$; whence $\tan a = \frac{a - bcc'}{c(a + b)}$.

5. *A sphere of radius a is supported on a rough inclined plane (for which the co-efficient of friction is c) by a string of length $\frac{a}{c}$, attached to it and to a point in the plane. Prove that the greatest possible elevation of the plane, in order that the sphere may rest when the string is a tangent is $2\tan^{-1}c$; and find the tension of the string and the pressure on the plane in the limiting position of equilibrum.*

Let 2θ be the angle between the string and the plane; a the inclination of the plane, and therefore the angle between the direction of the weight of the sphere (W) and the radius drawn to the point of contact; T the tension of the string and R the reaction of the plane.

Then $\sin \theta = \frac{c}{\sqrt{1 + c^2}}$, $\cos \theta = \frac{1}{\sqrt{1 + c^2}}$; $\therefore \sin 2\theta = \frac{2c}{1 + c^2}$, $\cos 2\theta = \frac{1 - c^2}{1 + c^2}$. Taking moments about centre of sphere, $T = cR$ (1). Resolving along and perpendicular to the plane $T\frac{1 - c^2}{1 + c^2} + cR = W \sin a$, (2); $T\frac{2c}{1 + c^2} + W \cos a = R$, (3). From (1) and (2) $T\frac{2}{1 + c^2} = W \sin a$, (4). From (1), (3) and (4) $T\frac{2c}{1 + c^2} + T\frac{2}{1 + c^2} \cot a = \frac{T}{c}$; whence $\cot a = \frac{1 - c^2}{2c}$, $\tan a = \frac{2c}{1 - c^2}$, $\tan \frac{1}{2}a = c$, or $a = 2\tan^{-1} c$. We shall find $T = cW$, and $R = W$.

The laws of friction above stated hold when there is sliding motion, although the friction is not of same amount as in the state bordering on motion; when there is a difference it is greater in the latter case than in the former. When there is sliding motion, the friction is independent of the velocity.

6. A body is projected up a rough inclined plane with velocity $2g$; the inclination of the plane to the horizon is 30°, and the co-efficient of friction is tan 15°. Find the distance along the plane which the body will describe.

The normal reaction of the plane is $W \cos 30°$, and $\therefore$ friction $= W \cos 30° \tan 15$; hence entire force down the plane $= W \sin 30° + W \cos 30° \tan 15°$. But acceleration $= \frac{\text{force}}{\text{mass}}$. Therefore acceleration down the plane $= (W \sin 30° + W \cos 30° \tan 15°) \div \frac{W}{g} = g(\sin 30° + \cos 30° \tan 15°) = g\frac{\sin 45°}{\cos 15°} = \frac{2g}{\sqrt{3} + 1}$. Now, if s be dis. described before body comes to rest, $v^2 = 2fs$; $\therefore (2g)^2 = 2 \cdot \frac{2g}{\sqrt{3} + 1} \cdot s$; $\therefore s = g(\sqrt{3} + 1)$.