APPLICATION OF NEWTON'S SECOND LAW OF MOTION TO CERTAIN HYDRAULIC PROBLEMS.*

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THE object of this paper is to discuss two phases of the application of Newton's second law of motion to hydraulic problems, the bearing of which is seldom clearly presented in textbooks on the subject. In order to do this it is necessary first to study that law in its most general form. The following statement of it, taken from Watson's "Physics," is a direct translation from Newton's works. "Change of motion is proportional to the impressed force and takes place in the direction of the straight line in which the force acts." As pointed out by Watson, the word "motion" as used by Newton was intended to convey the meaning "quantity of motion" or, as we speak of it, "momentum," *i.e.*, mass times velocity; also that the word "change" as used by Newton was intended to convey the meaning "rate of change." From the law as so interpreted, we may write at once its general differential equation form:

$$\frac{d\,(Mv)}{dt} \propto P, \quad i.e., \frac{(d\,Mv)}{dt} = kP \qquad (1)$$

and by general agreement as to choice of units, the value of the constant "k" has been taken as unity.

NOMENCLATURE.

d = symbol for differential quantity.	Q = rate of flow of liquid in units of volume per unit of time = $\frac{dV}{dt}$.
M = mass.	c = coefficient of contraction.
v = velocity.	r = radius of circular orifice.
t = time.	ϕ = coefficient of velocity, or friction for an orifice.
P = force.	a = acceleration = $\frac{dv}{dt}$.
k = a constant.	x = a rise in free water surface measured in units of length.
F = an area.	l = a length.
s = distance.	C = velocity of a wave front, or "bore."
γ = heaviness of liquid, or weight per unit of volume.	b = inside measurement of the width of an open flume.
g = acceleration due to gravity.	
p = pressure on unit area.	
p_a = atmospheric pressure on unit area.	
h = a depth of water measured in units of length.	
V = volume.	

Equation (1) therefore becomes on expanding the left-hand member and putting $k = 1$,

$$v\frac{dM}{dt} + M\frac{dv}{dt} = P \qquad (2)$$

The first phase of the application of this differential equation to problems in hydraulics mentioned at the beginning of this paper involves the interpretation of the term $v\frac{dM}{dt}$. Evidently it represents that momentum which is produced during each unit of time, dt, by imparting the finite velocity, v, to a small (differential) mass, dM; while the term $M\frac{dv}{dt}$, with which we are all so familiar in the mechanics of solids, represents that momentum which is produced during each unit of time, dt, by imparting the small (differential) velocity, dv, to the finite mass, M.

In order to show clearly the application of Equation (2) to a given problem, we will consider the discharge from a re-entrant mouthpiece, such as shown in Fig. 1. The vessel is considered to be very large compared with the mouthpiece, or orifice. We will first apply Equation (2) to the study of the motion of a definite element of the mass M, which is a prism of a stream line as indicated in the figure. It is readily seen that $M = \frac{F\,ds\gamma}{g}$ and P = Force in direction of motion = $-\,Fdp$. This force operates to impart a small increment of velocity to the total mass under consideration, during each unit of time dt, and we therefore write—

$$M\frac{dv}{dt} = -\,Fdp \text{ or } -\frac{g}{\gamma}\cdot\frac{dp}{ds} = \frac{dv}{dt}$$

$$-\frac{g}{\gamma}\cdot\frac{dp}{dv} = \frac{ds}{dt} = v$$

$$\int_0^v vdv = -\frac{g}{\gamma}\int_{h\gamma + p_a}^{p_a} dp$$

$$0 - \frac{v^2}{2} = -\frac{g}{\gamma}\Big[h\gamma + p_a - p_a\Big]$$

$$v = \sqrt{2\,gh} \qquad (3)$$

We will now apply Equation (2) to a study of the entire mass of water, both in the vessel and in the issuing jet. The re-entrant mouthpiece being of the greatest length which permits the jet to spring clear and not adhere to the sides, the velocity of approach near the points c and d is negligible and the pressure at those points may therefore be taken as hydro-static. The pressure on all sides of the entire mass of water is therefore balanced with the single exception of that area equal to and opposite the opening $A\,B$. If this area be designated by "F" and "h" be the head of water on the centre of the area under consideration, then the net unbalanced pressure or force is $Fh\gamma$, which becomes our "P" in equation (2). This force does not operate to impart a small increment of velocity to the total mass, M, during each unit of time, dt, but instead is continually imparting the velocity v, given by equation (3), to a small mass dM which issues from the vessel during each unit of time, dt. Therefore we write

$$v\frac{dM}{dt} = Fh\gamma$$

but

$$\frac{dM}{dt} = \frac{d\left(\frac{V\gamma}{g}\right)}{dt} = \frac{\gamma}{g}\cdot\frac{dV}{dt} = \frac{Q\gamma}{g}.$$

Hence

$$\frac{Qv\gamma}{g} = Fh\gamma \qquad (4)$$

If "c" is the coefficient of contraction $Q = cFv$ and we have

$$\frac{cFv^2\gamma}{g} = Fh\gamma \qquad (5)$$

and replacing v^2 by $2\,gh$, we obtain the equation

$$c = 0.50$$

That is, we have shown that the coefficient of contraction can be theoretically determined and explained by a general application of Newton's second law of motion to the problem involved.

The foregoing theoretical determination of the coefficient of contraction for a re-entrant mouth piece is given by a number of writers but so far as known, the following theoretical determination of the coefficient of contraction for a thin-edged orifice in the vertical side of a large vessel or tank is new.

In this case, shown in Fig. 2, the velocity near the points A and B is not negligible. It is specified that the orifice is small compared with the dimensions of the tank

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