Let $S_1 = \frac{1}{1^2} + \dots + \frac{1}{\overline{\Sigma n+1}^2}$.

$\therefore S_1 - \frac{1}{1^2} = \frac{1}{(1^2+2^2)} + \dots + \frac{1}{\Sigma n^2}$,

from the last two equations by subtraction we have $S = 1 - \frac{6}{(n+1)(n+2)(2n+3)}$.

VI. If n be prime, prove that any number in the scale whose radix is $2n$ ends in the same digit as its n^{th} power.

Last digit in number need only be considered; let it be r,

$r^{n-1} - 1$ is a multiple of n (Fermat).

$\therefore$ so is $r^n - r$.

Now if r be odd $r^{n-1} - 1$ is even, $\therefore r^n - r$ is a multiple of $2n$; if r be even, $r^n - r$ is even. Thus r^n is a multiple of $2n + r$,

$\therefore$ power ends in r.

VII. If

$\frac{p_r}{q_r}$ be the r^{th} convergent to $\frac{\sqrt{5}+1}{2}$

prove that

$$p_3 + p_5 + \dots + p_{2n-1} = p_{2n} - p_2,$$
$$q_3 + q_5 + \dots + q_{2n-1} = q_{2n} - q_2.$$

By Law of Formation—

$$p_{2n} = p_{2n-1} + p_{2n-2}$$
$$p_{2n-2} = p_{2n-3} + p_{2n-4}$$
$$\&c. = \&c.$$
$$p_4 = p_3 + p_2.$$

Cancel and add; treat second part in same way.

VIII. See Todhunter's Larger Algebra, Art. 499.

IX. If $\phi(r) = \underline{|n}$

$$\left\{\frac{1}{\underline{|r}\,\underline{|n-r}} + \frac{1}{\underline{|r-1}\,\underline{|n-r+1}} \cdot r + \frac{1}{\underline{|r-2}\,\underline{|n-r+2}}\,\frac{(r-1)(r-2)}{1\cdot 2} + \dots\right\}$$

prove that

$$2\left[\phi(0) + \phi(1) + \dots + \phi(n-1)\right] + \phi(n) = 3^n.$$

Question should read

Prove that $\phi(0) + \phi(1) + \dots + \phi(n) = 3^n$.

We have $\phi(r)$ equals coefficient of x^r in

$$\frac{n(n-1)\dots(n-r+1)}{\underline{|r}} x^r (1+x)^{r+1} + \frac{n(n-1)\dots(n-r+2)}{\underline{|r-1}} x^{r-1}(1-x)^r + \dots$$

i.e., in $(1+x)\left\{1 + x\,\overline{1+x}\right\}^n$. Find sum of coefficients in this by putting $x=1$.

X. Eliminate x, y, z from the simultaneous equations

$$\begin{cases} \dfrac{a}{x} = \dfrac{1}{y} + \dfrac{1}{z} \\ \dfrac{\beta}{y} = \dfrac{1}{z} + \dfrac{1}{x} \\ \dfrac{\gamma}{z} = \dfrac{1}{x} + \dfrac{1}{y} \end{cases}$$

Why are these three equations sufficient for the elimination of the three unknowns? The above may be solved as under.

$$\begin{vmatrix} -a, & 1, & 1 \\ 1, & -\beta, & 1 \\ 1, & 1, & -\gamma \end{vmatrix} = 0.$$

11. If $A + B + C = \frac{\pi}{2}$, shew that

(1) $\cot A + \cot B + \cot C = \cot A \cot B \cot C$.

(2) $\tan A + \tan B + \tan C = \tan A \tan B \tan C + \sec A \sec B \sec C$.

(1) and (2) are obtained by expanding $\cos(A+B+C) = 0$, and $\sin(A+B+C) = 1$, and dividing their expansions by $\sin A \sin B \sin C$, $\cos A \cos B \cos C$ respectively.

13. On the side BC of the triangle ABC are drawn two equilateral triangles, $A'BC$ and $A''BC$; likewise, the equilateral triangle $B'CA$, $B''CA$ and $C'AB$, $C''AB$ are drawn on the sides CA and AB respectively, Prove that

$$A'A \cdot AA'' = B'B \cdot BB'' = C'C \cdot CC''.$$

$$AA'^2 = a^2 + c^2 - 2ac\cos\left(B - \frac{\pi}{3}\right)$$

$$AA''^2 = a^2 + c^2 - 2ac\cos\left(B + \frac{\pi}{3}\right),$$