$$M = \int\int p' y' du' dy'. \quad \text{(Eq. 1.)}$$

Now, let us consider a unit section of fibres situated at unit distance from the neutral surface. The extension or compression of fibres per unit of length being c, the force producing it is $E c = p =$ intensity of stress at unit's distance. According to the common theory of flexure, the intensities are directly proportional to the distances from the neutral surface, therefore $p' : p :: y' : 1$ or $p' = p y' = E c y'$. Substituting this in Eq. 1., gives $M = \int\int E c y'^2 du' dy' = E c \int\int y'^2 du' dy' = E c I$ and $c = \frac{M}{E I}$

Now, remembering that in actual practice the curvature of a beam is very slight, let us consider an indefinitely short portion of the neutral surface, the length of which is dx; the strain for such a length at unit's distance would be $c\ dx$.

In Fig. 2, which is, of course, greatly exaggerated, A B = dx, C is the centre of curvature of the beam, A D is equal to unity, therefore F E = $c\ dx$, B K and A L are tangents to the neutral surface at the points B and A respectively, G H is drawn parallel to D C, K M perpendicular to A L and L N is a vertical line through the origin O. B O being a finite distance, while A B is indefinitely small, and the curvature being so very slight we can say without any sensible error that A B K is a straight line equal in length to A M, or the curve A B O which is also equal to A K or x. The two triangles E B F and M A K are similar, the sides being mutually perpendicular, therefore,

F E : F B :: K M : M A or $c\ dx : 1 ::$ K M : x,
or K M = $x\ c\ dx$.

But K M does not differ sensibly from K L, therefore K L = $x\ c\ dx$. But the summation of the consecutive values of K L between O and A is equal to the distance O L = D, which we may term the vertical deflection of the tangent at the point A (not necessarily equal to the deflection of the beam in the ordinary acceptance of the term).

We may now write the following equations which are true under the assumed restrictions.

$$S = \Sigma P \quad \text{[Eq. 2.]}$$
$$M = \Sigma P z \quad \text{[Eq. 3.]}$$
$$C = \frac{M}{E I} \quad \text{[Eq. 4.]}$$
$$D = \Sigma x\ c\ dx = \Sigma \frac{x M}{E I} dx. \quad \text{[Eq. 5.]}$$

Let us next investigate some general considerations in regard to a portion of a continuous beam between two consecutive points of suppoot.

If the beam were simply supported at the ends, the reactions at those points could be ascertained by applying the principle of the lever; but, if one or both ends are not simply supported, the reactions will differ from those found according to that law; this difference may be accounted for by supposing a portion of the reaction at one end to be transferred to the other, by means of the application of a couple whose lever arm is equal to the length of span between the points of support, and whose forces are each equal to the difference between the actual reaction at one end and the reaction as calculated by the law of the lever.

As this change of reaction is caused by a partial fixing of the end or ends of the beam, it is evident that the bending of the supposed couple will be of an opposite kind to that which exists in a beam simply supported at the ends.

In addition to this couple, we can suppose two equal and opposite couples, applied to the beam at each end, which balance each other by means of the beam, thus subjecting it to another moment, without at all affecting the reactions. For example, in a bridge truss we might apply a certain tensile force at each end of the top chord, and an equal compressive force at each end of the bottom chord, producing a constant moment throughout the span equal to the product of one of the forces by the depth of the truss, without at all affecting the reactions due to the loads upon the span. A practical instance of this would be the case of three continuous spans, the outer ones being equal in length and similarly loaded.

In Fig. 3, let A and B be the supports of the beam A B, which is loaded at K by the weight P, giving the reactions A R and B R'. R F = R' F' is the force of the couple, making A F and B F' the reactions after the couple is applied. For the reactions A R and B R' the ordinates parallel to C K in the triangle A C B give the bending moments due to the weight P.

Let the ordinates parallel to A H in the rectangle A H Q B represent, according to the same scale, the constant moment due to the balanced couples; and let the ordinates in the triangle Q H G represent the moments of the force R' F'. The resultant moment at any point will be the algebraic sum of the ordinates in the rectangle and the two triangles.

Take the origin of coordinates at A, and let x be measured horizontally towards B.

Let G A = M_a and B Q = M_b. Join A Q.

Then the moment at any point S is equal to

$$+ S T - S U - U W =$$

$$+ M_1 - M_b \frac{x}{l} - M_a \frac{l - x}{l} = M. \quad \text{[Eq. 6.]}$$

It is to be noticed that ordinates above A B are positive and tend to bow the beam downward at the centre, while the ordinates below A B are negative and tend to bow the beam upward at the centre. In Fig. 3, the moment produced by the balanced couples was assumed to act in conjunction with the moment of the single couple. Had these moments acted in opposite directions, as in Fig, 4, where A Z, the difference between A G and A H, is M_a, we would have had

$$M = + S T - U Y + S U,$$

$$\text{or } M = + M_1 - M_a \frac{l - x}{l} + M_b \frac{x}{l} \quad \text{[Eq. 7.]}$$

Thus we see that the second members of Eqs. 6 and 7 differ only in the signs of the terms, and if we consider the signs inherent we may write them both

$$M = M_1 + M_a \frac{l - x}{l} + M_b \frac{x}{l} \quad \text{[Eq. 8.]}$$