Let y be numerically less than 1 so that we may expand $(1+y)^x$ by the binomial theorem. Then

$$(1+y)^x = 1 + x.y + \frac{x(x-1)}{1.2}y^2 + \ldots + \frac{x(x-1)\ldots(x-r+1)}{1.2\ldots r}y^r + \ldots$$

Then equating the coefficient of x in the two values of $(1+y)^x$, we have

$$\log_e(1+y) = \frac{y}{1} - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots\ldots$$

This is valid for values of y numerically less than 1. Replacing y by x as the theorem is generally quoted in terms of x, we have for $|x| < 1$

$$\log_e(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$$

This is the **logarithmic theorem.**

It can be shewn to be valid for $x = 1$, though not for $x = -1$. By giving to x the value $-\frac{1}{2}$ we can find to any degree of accuracy $\log_e 2$ and *then* by putting $x = -\frac{1}{3}$ the value of $\log_e 3$ and so on. This is not the best way of finding these logarithms, as the series may be modified to give the approximate values more readily, but this matter we shall not deal with further.

When the logarithms of successive integers to the base e have been found they can be changed to logarithms to base 10. For having found $\log_e 10$ we have for any number N *

$$\log_{10} N = \frac{\log_e N}{\log_e 10}.$$