60. Prove sum of products of 1st n natural numbers taken two and two together

$$=\frac{n.(n-1).(n+1).(3n+2)}{24}$$

We have, denoting required sum by S, $(a+b+c+d+\ldots\text{to } n \text{ terms})^2=a^2+b^2+c^2+\ldots$ to n terms$+2(ab+ac+\ldots)$.

Put $a=1$, $b=2$, $c=3$, etc.,

$$\therefore 2S=\left(\frac{n.\overline{n+1}}{2}\right)^2-\frac{n.(n+1).(2n+1)}{2.3}.$$

$$\therefore S=\frac{(n-1).n.(n+1).(3n+2)}{24}.$$

61. Shew that the remainder after n terms of the expansion of $(1-x)^{-2}$ is

$$\frac{(n+1)x^n-nx^{n+1}}{(1-x)^2}$$

$$(1-x)^{-2}=1+2x+3x^2+\ldots+(n+1)x^n+\ldots$$

Let $S=1+2x+3x^2+\ldots+nx^{n-1}$,

$$\therefore Sx=x+2x^2+\ldots+\overline{n-1}\,x^{n-1}+nx^n,$$

$$\therefore \quad S=\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x},$$

$$\therefore \text{remainder}=(1-x)^{-2}-S$$

$$=\frac{(n+1)x^n-nx^{n+1}}{(1-x)^2}.$$

62. The sum of the first $\overline{r+1}$ coefficients of the expansion of $(1-x)^{-m}$ is $\dfrac{\underline{|m+r}}{\underline{|m}\ \underline{|r}}$, m being a positive integer.

$$(1-x)^{-m}=1+c_1x+c_2x^2+\ldots+c_rx^r+\ldots$$

$$(1-x)^{-1}=1+x+x^2+\ldots+x^r+\ldots$$

$\therefore$ required sum equals coefficient of x^r in $(1-x)^{\overline{-m+1}}$.

$$=\frac{(m+1)(m+2)\ldots(m+r)}{\underline{|r}}.$$

$$=\frac{\underline{|m+r}}{\underline{|m}\ \underline{|r}}.$$

67. Three circles are so inscribed in a triangle that each touches the other two and two sides of the triangle; prove radius of that which touches sides AB, AC is

$$\frac{r}{2}\left\{\frac{\left(1+\tan\frac{B}{4}\right)\left(1+\tan\frac{C}{4}\right)}{\left(1+\tan\frac{A}{4}\right)}\right\}.$$

Let circles whose centres are O_3 O_2 touch side BC in a_2, a_1 respectively, and also touch each other and sides AC, AB respectively, and circle centre O_1 touch the other two circles and sides AB, AC.

Then r, r_1, r_2, r_3, denoting radii of inscribed circle of triangle ABC, and of above circles, we have

$$O_2O_3=r_2+r_3,\sqrt{\{(r_2+r_3)^2-(r_2-r_3)^2\}}$$

$$=a_1a_2=2\sqrt{r_2r_3},$$

$$\therefore BC=a=r_2\cot\frac{B}{2}+r_3\cot\frac{C}{2}+2\sqrt{r_2r_3},$$

$$=r\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right).$$

anal$=$anal.

$$\therefore r_3\cot\frac{C}{2}+r_1\cot\frac{A}{2}+2\sqrt{r_1r_3}$$

$$=r\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right) \quad (1)$$

anal$=$anal.

$$\text{whence } \sqrt{r_1}\cos\frac{A}{2}+\sqrt{r_3}\sin\frac{A}{2}$$

$$=\sqrt{r_2}\cos\frac{B}{2}+\sqrt{r_3}\sin\frac{B}{2}$$

anal$=$anal.

$$\text{whence } \sqrt{r_1}\left(\cos\frac{A}{2}+\sin\frac{B}{2}-\sin\frac{C}{2}\right)$$

$$=\sqrt{r_3}\left(\cos\frac{C}{2}+\sin\frac{B}{2}-\sin\frac{A}{2}\right).$$

$$\therefore \frac{r_1}{r_3}=\left\{\frac{1+\tan\frac{C}{4}}{1+\tan\frac{A}{4}}\right\}^2$$

Substituting in (1) and reducing, required expression is obtained.

68. If $\dfrac{a+cx}{c+ax}$ be expanded in a series ascending by powers of $(1-x)$ and $(1+x)$, and A, B be the coefficients of $(1-x)^{n+1}$ and $(1+x)^{n+1}$ respectively; then

$$\frac{A}{B}=\pm\left(\frac{c-a}{c+a}\right)^{n+1}$$

$\pm$ according as n is even or odd.

$$\frac{a+cx}{c+ax}$$

$$=\left(\frac{c+ax}{a+cx}\right)^{-1}=\left\{1+\frac{(c-a)(1-x)}{a+cx}\right\}^{-1} \quad (1)$$

$$=\left(\frac{c+ax}{a+xc}\right)^{-1}=-\left\{1-\frac{(c+a)(1+x)}{a+cx}\right\}^{-1} \quad (2)$$

Expanding the above by Binomial Theorem we obtain required result.