K, K', K'' and K''' being rational. But the terms A_1, A_2, A_4, A_3 circulate with Δ_1, Δ_2, Δ_4, Δ_3. Therefore

$$A_2 = K + K'\,\Delta_2 + K''\,\Delta_3 + K'''\,\Delta_2\,\Delta_3,$$
$$A_4 = K + K'\,\Delta_4 + K''\,\Delta_1 + K'''\,\Delta_1\,\Delta_4,$$
$$A_3 = K + K'\,\Delta_3 + K''\,\Delta_2 + K'''\,\Delta_2\,\Delta_3,$$

These are Abel's values.

§72. Keeping in view the values of Δ_1, Δ_2, etc., in (67), and also that $z = 1 + e^2$, and $s = hz + h\sqrt{z}$, any rational values that may be assigned to m, n, e, h, K, K', K'' and K''' make r_1, as presented in (74), the root of an equation of the fifth degree. For, any rational values of m, n, etc., make the values of S_1, S_2, etc., in §62, rational.

§73. It may be noted that, not only is the expression for r_1 in (74) the root of a quintic equation whose auxiliary biquadratic is irreducible, but on the understanding that the surds $\sqrt{s}$ and $\sqrt{z}$ in Δ_1 may be reducible, the expression for r_1 in (74) contains the roots both of all equations of the fifth degree whose auxiliary biquadratics have their roots rational, and of all that have quadratic sub-auxiliaries. It is unnecessary to offer proof of this.

§74. The equation $x^5 - 10x^3 + 5x^2 + 10x + 1 = 0$ is an example of a solvable quintic with its auxiliary biquadratic irreducible. One of its roots is

$$\omega^{\frac{1}{5}} + \omega\omega^{\frac{2}{5}} + \omega^3\omega^{\frac{3}{5}} + \omega^4\omega^{\frac{4}{5}},$$

ω being a primitive fifth root of unity. It is obvious that this root satisfies all the conditions that have been pointed out in the preceding analysis as necessary. A root of an equation of the seventh degree of the same character is

$$\omega^{\frac{1}{7}} + \omega^4\omega^{\frac{2}{7}} + \omega^4\omega^{\frac{3}{7}} + \omega^2\omega^{\frac{4}{7}} + \omega^3\omega^{\frac{5}{7}} + \omega^6\omega^{\frac{6}{7}},$$

ω being a primitive seventh root of unity. The general form under which these instances fall can readily be found. Take the cycle that contains all the primitive $(m^2)^{\text{th}}$ roots of unity,

$$\theta,\ \theta^{\beta},\ \theta^{\beta^2},\ \text{etc.} \tag{75}$$

m being prime. The number of terms in the cycle is $(m-1)^2$. Let θ_1 be the $(m+1)^{\text{th}}$ term in the cycle (75), θ_2 the $(2m+1)^{\text{th}}$ term, and so on. Then the root of an equation of the m^{th} degree, including the instances above given, is

$$r_1 = (\theta + \theta^{-1}) + (\theta_1 + \theta_1^{-1}) + \ldots + \left(\theta_{\frac{m-3}{2}} + \theta_{\frac{m-3}{2}}^{-1}\right).$$