(3) and (5) subsist together. In k_1, by the Corollary in §4, we can change w into w^e. Therefore

$$k_c = w^{ac}(A_{cv}A_c^{-v})\,Q\,(q_c q'_c q''_c \ldots .).$$

By comparing this with (135), $w^r = w^{ac}$. Therefore the first of equations (127) becomes

$$R_{cv}^{\frac{1}{n}} = w^{ac}A_{cv}\,Q\,(P_{cvm}^{m}\phi_{cv\sigma}^{\sigma} \ldots .)^{\frac{1}{n}}.$$

Replacing Q by $(F_{cv\beta}^{\beta} \ldots .)^{\frac{1}{n}}$, and putting e for c, which we are entitled to do because w^e may be any one of the roots included in the general form w^e,

$$R_{ev}^{\frac{1}{n}} = w^{ea}A_{ev}\,(P_{evm}^{m}\phi_{ev\sigma}^{\sigma} \ldots . F_{ev\beta}^{\beta})^{\frac{1}{n}},$$

which is the form of $R_{ev}^{\frac{1}{n}}$ in (110).

Sufficiency of the Forms.

§57. Here we assume that R_1 has the form (104), and that the forms in (111) are determined by the equations (109), (110), etc., while $R_0^{\frac{1}{n}}$ receives its rational value; and we have to prove that the expression (105) is the root of a pure uni-serial Abelian equation of the n^{th} degree, provided always that the equation of the n^{th} degree, of which it is the root, is irreducible. *In the first place*, it has been shown that there is an n^{th} root of R_0 which has a rational value; and, by hypothesis, $R_0^{\frac{1}{n}}$ has been taken with this rational value. *In the second place*, an equation of the type (3) subsists for every integral value of z. For

$$R_z R_1^{-z} = (A_z A_1^{-z})^n (P_{mz}^{m} P_m^{-mz})(\phi_{\sigma z}^{\sigma}\phi_\sigma^{-\sigma z}) \ldots . (F_{\beta z}^{\beta} F_\beta^{-\beta z}).$$

But P_m is of the form of the fundamental element of the root of a pure uni-serial Abelian quartic. Therefore, by §5, $P_{mz}P_m^{-z}$ is the fourth power of a rational function of the primitive fourth root of unity w^m. Therefore, because $n = 4m$, $P_{mz}^{m}P_m^{-mz}$ is the n^{th} power of a rational function of w. Also, it can be proved, exactly as in §44, that, whether z be a multiple of s or not, $\phi_{\sigma z}^{\sigma}\phi_\sigma^{-\sigma z}$ is the n^{th} power of a rational function of w. And so of the other corresponding expressions. Therefore $R_z R_1^{-z}$ is the n^{th} power of a rational function of w. *In the third place*, we have to show that an equation such as (5) subsists for every corresponding equation (3). For, let z be prime to n. It is then included in e. Also, since z and e are both prime to n, ze is included in e; and unity is included in e. But, from the manner in which the root was constructed from its fundamental element, $R_e^{\frac{1}{n}}$ is determined as in (109). Therefore