SOLUTION—In placing successively $x+y=t$, $xy=t^1$, $t+t^1=n$, $tt^1=n^1$, $n+n^1=u$, $nn^1=u^1$, we obtain $u+u^1=a$, and $uu^1=b$, which shows that u and u^1 are the roots of the equation $U^2-aU+b=0$; u and u^1 being known, we have n and n^1 by the equation, $V^2-uV+u^1=0$. Then we would have t and t^1 by the relation $T^2-nT+n^1=0$, and finally $X^2-tX+t^1=0$ would give x and y.

6. Two trains, a Passenger and Freight, leave the southern end of a railroad and travel north at the same instant that an Express leaves the nothern end of the same road going south. When the Passenger train arrives at middle station, the Freight is midway beween the Passenger and Express. The Passenger train then moves on 60 miles when it is as far from the middle station as the Freight, and the Express has finished the journey: here an accident happens that delays the Passenger 10 hours, when it proceeds on its journey and arrives at the destination at the same time as the Freight. Required the rate of each, and the length of the road.

SOLUTION—Let t=the time it takes the Passenger to make half the journey, x=the rate of the Passenger, y=the rate of the Freight, and z=that of the Express. Then $2tx$=the length of the road, $tx-ty$=the distance of the Freight from the Passenger when the Passenger arrives at the middle station, and $tz-tx$=the distance of the Express from the Passenger; hence, since the Freight is midway between the Passenger and the Express, we have –

$$tz-tx=2(tx-ty), \text{ or } 3x=2y+z \ldots\ldots(1).$$

When the Passenger is 60 miles past the middle station, the distance of Freight from that station is $tx+ty-(60y\div x)=60\,..(2)$, and since the distance traveled by the Express at that time=the sum of the distances traveled by the other two trains, we have $z=x+y\,..(3)$. The difference of the time required for the Passenger and Freight to make the journey is $(2tx\div y)-2t=10\,..(4)$.

Subtracting (2) × 2 from (4) × y, we have $xy=12(x+y)\,..(5)$. From (1) and (3) $y=\frac{2}{3}x$. Substituting this value of y in (5), we find x=30 miles, y=20 miles, z=50 miles, and $2tx=10xy\div(x-y)=600$ miles.

7. A debt of $10000 is to be paid in 10 equal instalments, with interest at 8 per cent. per annum compounded every instant. Three of the equal payments are made before any interest accrues. An instalment is then paid at the end of each year until the year before the last, when no payment is made: two instalments being paid at the end of the last year. Required one of the equal payments.

SOLUTION—Let a=$10000, the debt, r=8, per cent., t=7 years, the time in which the debt was paid, x=one of the equal payments, and let each year be divided into n equal intervals.

Then if the interest is compounded at the end of each interval, the amount of the debt a at the end of t years is $a\left(1+\frac{r}{n}\right)^{nt}$

which developed by the Binomial Formula,

$$=a\left\{1+nt\left(\frac{r}{n}\right)+\frac{nt(nt-1)}{1.2}\left(\frac{r}{n}\right)^2+\frac{nt(nt-1)(nt-2)}{1.2.3}\left(\frac{r}{n}\right)^3+\ldots\right\}$$

$$=a\left\{1+tr+\frac{t(t-\frac{1}{n})}{1.2}r^2+\frac{t(t-\frac{1}{n})(t-\frac{2}{n})}{1.2.3}r^3+\ldots\right\}$$

But when the interest is compounded every instant, n is infinitely large, and hence each of the fractions $\frac{1}{n}, \frac{2}{n}, \ldots\ldots$ is equal to O, and the above series becomes

$$a\left(1+tr+\frac{t^2r^2}{1.2}+\frac{t^3r^3}{1.2.3}+\ldots\ldots\right)=ae^{tr}=ae^{7r},$$

where e=2.71828128, the Naperian base of logarithms. The amount of the three payments made at first is $3xe^{7r}$; that of the payment made at the end of the second year is xe^{6r}; and so on. Now since the sum of the amounts of the payments must be equal to the amount of the debt, we have

$$x(3e^{7r}+e^{6r}+e^{5r}+e^{4r}+e^{3r}+e^{2r}+2)=ae^{7r},$$

$$\text{whence } x=\frac{a(e^{8r}-e^{7r})}{3e^{8r}-2e^{7r}-e^{2r}+2e^{r}-2}=\$1234.50$$

8. Find the par value of $11000 debentures issued for 25 years, interest payable @ 6% per annum; *i.e.* 3% half yearly. Money worth 7% per annum.

SOLUTION—At the end of 25 years the buyer will have received the following sums:—

Capital repaid					=	$11000
49th and last payment of interest					=	330
48th payment—in hand half a year—value					=	330(1·035)
47th	"	"	one	" "	=	$330(1·035)^2$
46th	"	"	$1\frac{1}{2}$	" "	=	$330(1·035)^3$
&c.			&c.		=	&c.
1st	"	"	$24\frac{1}{2}$	" "	=	$330(1·035)^{49}$

Total product of debentures in 25 years.

$=\$11000+330+330(1·035)+330(1·035)^2+\&c.+330(1·035)^{49}$

$=11000+330\{1+1·035+1·035^2+\ldots\ldots1·025^{49}\}$

$=11000+330\times\frac{1·035^{50}-1}{1·035-1}$

Now log. $1·035^{50}$=50 log. 1·035=50 × ·0149403=·747015 =log. 5·58491.

∴ prod. of deben.=11000+(330 × 4·58491) ÷ ·035=$53980·058.

We have now to find the present worth of this in 25 yrs @7% per an.

Present worth=53980·058 ÷ $(1·07)^{25}$.

Now log. $1·07^{25}$=25 log. 1·07,

=25 × ·0293838=·734595=log. 5·42745,

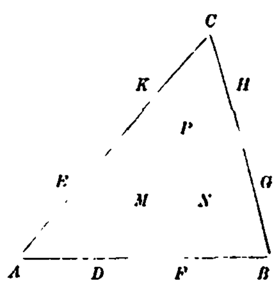
And 53980 058 ÷ 5·42745=$9945·749.

Hence $1100 debentures=$9945·749,

i.e. $1 " = ·9041+ or 90½ nearly.

9. The sides of a triangle being 130, 140, and 150, what are the radii of three circles so inscribed that each of them touches the other two and two sides of the triangle?

SOLUTION BY PROF. SEITZ, Missouri Normal Schools.



Let ABC be the triangle, M, N, P, the centres of the circles, D, E, F, G, H, K, the points of tangency. Let $MD=x$, $NF=y$, $PH=z$, r=radius of the inscribed circle of the triangle ABC, $BC=a=130$, $CA=b=140$, and $AB=c=150$. Then we have $AD=AE=x\cot\frac{1}{2}A$, $BF=BG=y\cot\frac{1}{2}B$, $CH=CK=z\cot\frac{1}{2}C$, $DF=2\sqrt{(xy)}$, $GH=2\sqrt{(yz)}$, $EK=2\sqrt{(xz)}$, $r=40$, and we have the equations,

$x\cot\frac{1}{2}A+2\sqrt{(xy)}+y\cot\frac{1}{2}B=c\,..(1)$

$x\cot\frac{1}{2}A+2\sqrt{(xz)}+z\cot\frac{1}{2}C=b\,..(2)$ $y\cot\frac{1}{2}B+2\sqrt{(yz)}+z\cot\frac{1}{2}C=a\,..(3)$

By trigonometry we have

$$\frac{\sin\frac{1}{2}B\cos\frac{1}{2}B}{b}=\frac{\sin\frac{1}{2}C\cos\frac{1}{2}C}{c}\,..(4), \text{ and } \frac{\sin\frac{1}{2}A\cos\frac{1}{2}A}{a}=\frac{\sin\frac{1}{2}C\cos\frac{1}{2}C}{c}\,..(5);$$

and from these two equations we can deduce the following,

$b(\cot\frac{1}{2}A-\tan\frac{1}{2}B)=c(\cot\frac{1}{2}A-\tan\frac{1}{2}C)\,..(6)$, and

$a(\cot\frac{1}{2}B-\tan\frac{1}{2}A)=c(\cot\frac{1}{2}B-\tan\frac{1}{2}C)\,..(7)$.

Dividing (1) by (2) and (3) respectively, and clearing of fractions, we have

$b[x\cot\frac{1}{2}A+2\sqrt{(xy)}+y\cot\frac{1}{2}B]=c[x\cot\frac{1}{2}A+2\sqrt{(xz)}+z\cot\frac{1}{2}C]\,..(8)$,

and $a[x\cot\frac{1}{2}A+2\sqrt{(xy)}+y\cot\frac{1}{2}B]=c[y\cot\frac{1}{2}B+2\sqrt{(yz)}+z\cot\frac{1}{2}C]\,..(9)$.

Subtracting (6) × x from (8), and (7) × y from (9), we have

$b[x\tan\frac{1}{2}B+2\sqrt{(xy)}+y\cot\frac{1}{2}B]=c[x\tan\frac{1}{2}C+2\sqrt{(xz)}+z\cot\frac{1}{2}C]\,..(10)$,

$a[x\cot\frac{1}{2}A+2\sqrt{(xy)}+y\tan\frac{1}{2}A]=c[y\tan\frac{1}{2}C+2\sqrt{(yz)}+z\cot\frac{1}{2}C]\,..(11)$.

Multiplying (10) by (4), and (11) by (5), and extracting the square root, we have

$\sqrt{x}\sin\frac{1}{2}B+\sqrt{y}\cos\frac{1}{2}B=\sqrt{x}\sin\frac{1}{2}C+\sqrt{z}\cos\frac{1}{2}C\,..(12)$, and

$\sqrt{x}\cos\frac{1}{2}A+\sqrt{y}\sin\frac{1}{2}A=\sqrt{y}\sin\frac{1}{2}C+\sqrt{z}\cos\frac{1}{2}C\,..(13)$.

Subtracting (13) from (12), we find

$$\frac{\sqrt{y}}{\sqrt{x}}=\frac{\sin\frac{1}{2}C+\cos\frac{1}{2}A-\sin\frac{1}{2}B}{\sin\frac{1}{2}C-\sin\frac{1}{2}A+\cos\frac{1}{2}B}=\frac{\cos\frac{1}{4}C+\cos\frac{1}{4}(2B+C)}{\cos\frac{1}{4}C+\cos\frac{1}{4}(2A+C)}=\frac{\cos\frac{1}{4}B\cos\frac{1}{4}(\pi-A)}{\cos\frac{1}{4}A\cos\frac{1}{4}(\pi-B)}$$

$$=\frac{1+\tan\frac{1}{4}A}{1+\tan\frac{1}{4}B}\,..(14). \quad \text{Similarly } \frac{\sqrt{z}}{\sqrt{x}}=\frac{1+\tan\frac{1}{4}A}{1+\tan\frac{1}{4}C}\,..(15).$$

Substituting the value of $\sqrt{y}$ from (14) in (1), we have

$$\left\{\cot\tfrac{1}{2}A+2\left(\frac{1+\tan\frac{1}{4}A}{1+\tan\frac{1}{4}B}\right)+\cot\tfrac{1}{2}B\left(\frac{1+\tan\frac{1}{4}A}{1+\tan\frac{1}{4}B}\right)^2\right\}x=c,$$

$$\text{or}\left\{\frac{1-\tan^2\frac{1}{4}A}{2\tan\frac{1}{4}A}+2\left(\frac{1+\tan\frac{1}{4}A}{1+\tan\frac{1}{4}B}\right)+\frac{1-\tan^2\frac{1}{4}B}{2\tan\frac{1}{4}B}\left(\frac{1+\tan\frac{1}{4}A}{1+\tan\frac{1}{4}B}\right)^2\right\}x=c,$$

whence

$$x=\frac{2c\tan\frac{1}{4}A\tan\frac{1}{4}B(1+\tan\frac{1}{4}B)}{(1+\tan\frac{1}{4}A)(\tan\frac{1}{4}A+\tan\frac{1}{4}B)(1-\tan\frac{1}{4}A\tan\frac{1}{4}B)[1+\tan\frac{1}{4}(A+B)]}$$

$$=\frac{c\sin\frac{1}{2}A\sin\frac{1}{2}B(1+\tan\frac{1}{4}B)}{\sin\frac{1}{2}(A+B)(1+\tan\frac{1}{4}A)[1+\tan\frac{1}{4}(\pi-C)]}=\frac{r(1+\tan\frac{1}{4}B)}{(1+\tan\frac{1}{4}A)[1+\tan\frac{1}{4}(\pi-C)]}$$

$$=\frac{\frac{1}{2}r(1+\tan\frac{1}{4}B)(1+\tan\frac{1}{4}C)}{1+\tan\frac{1}{4}A}=\tfrac{5}{2}(17+4\sqrt{5}-2\sqrt{13}-\sqrt{65})=26·677279.$$

Substituting the value of x in (14) and (15), we find

$$y=\frac{\frac{1}{2}r(1+\tan\frac{1}{4}A)(1+\tan\frac{1}{4}C)}{1+\tan\frac{1}{4}B}=\tfrac{20}{7}(17-4\sqrt{5}-2\sqrt{13}+\sqrt{65})=25·44823$$

$$\text{and } z=\frac{\frac{1}{2}r(1+\tan\frac{1}{4}A)(1+\tan\frac{1}{4}B)}{1+\tan\frac{1}{4}C}=\tfrac{10}{3}(17-4\sqrt{5}+2\sqrt{13}-\sqrt{65})=24·015.$$