CERTAIN PROPERTIES OF BERNOULLI'S TABLES.

I.

TO OBTAIN ANY RATIO BETWEEN TWO SERIES.

Any desirable ratio between the sums of the terms of different series framed according to Bernoulli's Table can be secured by means of the following deduction from the 12th property.

Let S = the sum of n terms in any column a of the table.
" l = the last of n terms in column a

Then

$$S : n.l = 1 : a$$

or

$$S.a = n.l$$

A deduction from Formula No. (1) given in Bernoulli's Table.

EXAMPLE.—The sum of 19 terms in Column XI is 75,582
The last or 19th term in Column XI is 43,758
Then 75,582 is to $19 \times 43{,}758$ as 1 : 11
or $11 \times 75{,}582 = 19 \times 43{,}758$

$$\begin{array}{r} 75{,}582 \\ 11 \\ \hline 75{,}582 \\ 75{,}582 \\ \hline 831{,}402 \end{array} \qquad\qquad \begin{array}{r} 43{,}758 \\ 19 \\ \hline 393{,}822 \\ 43{,}758 \\ \hline 831{,}402 \end{array}$$

Bernoulli expresses this remarkable relation in the following quaint language, as given in the translation published by FRANCIS MASERES, Cursitor Baron of the Court of Exchequer in 1795: "The sum of any number of terms in any of the vertical columns contained in the foregoing table of combinations is to the sum of the same number of terms all equal to the last of them, in the proportion of 1 to the exponent of the said column, or to the number which denotes or expresses its place in the said table."

II.

GENERAL EXPRESSION FOR THE TERMS OF ANY ONE OF THE VERTICAL COLUMNS IN BERNOULLI'S TABLE.

Let S = the sum of the Series in any Column a to n terms, including cyphers, according to Bernoulli's Table. Then

$$S = 1 + a + \frac{a(a+1)}{2} + \frac{a(a+1)(a+2)}{2.3} + \frac{a(a+1)(a+2)(a+3)}{2,3,4} + \frac{a(a+1)(a+2)(a+3)(a+4)}{2,3,4,5} + \text{\&c., \&c.}$$

EXAMPLE.—Let $a = 12$, and n the number of terms = 21; equal to $10 + 11 = 21$, including cyphers, being 10 quantities and 11 cyphers.

The Series is

$$S = 1 + 12 + 78 + 364 + 1365 + 4368 + 12{,}376 + 31{,}824 + 75{,}582 + 167{,}960$$

And according to Bernoulli's formula,

$$S = \frac{l \times n}{a} \qquad S = \frac{167{,}960 \times 21}{12} = 293{,}930$$

In detail, the calculation is as follows:

$a = 12$; n = 10 quantities, then,

$$S = 1 + 12 + \frac{12,13}{2} + \frac{12,13,14}{2,3} + \frac{12,13,14,15}{2,3,4} + \frac{12,13,14,15,16}{2,3,4,5} + \frac{12,13,14,15,16,17}{2,3,4,5,6} + \frac{12,13,14,15,16,17,18}{2,3,4,5,6,7}$$
$$+ \frac{12,13,14,15,16,17,18,19}{2,3,4,5,6,7,8} + \frac{12,13,14,15,16,17,18,19,20}{2,3,4,5,6,7,8,9}$$

$= 1 + 12 + 78 + 364 + 1365 + 4386 + 12{,}376 + 31{,}824 + 75{,}582 + 167{,}960$, which is the XIIth Column in Bernoulli's Table to 21 terms, and each of the quantities in the series is equal to the sum of the series in the column next preceding it to the left and beginning with the quantity one square above it. Thus, 75,582 is the sum of all the quantities in Column XI from 43,758 upwards, and there are 19 terms in that column, including cyphers; or 75,582 is the sum of the VIIIth column, beginning with 31,824 and thence upwards. 31,824 is the sum of the VIIth column, beginning with 12,376 and thence upwards; or of the XIth column, beginning with 19,448 and thence upwards, and so on for all the quantities in the series.

It is to be noticed that the sloping column to the left of 75,582 up to No. 11, consists of the same figures as the vertical column No. XII over 75,582. This rule holds good throughout, together with numerous other relations between columns and parts of columns, which it is not necessary now to point out.