cation a group of units known as the *multiplicand* is repeated a number of times expressed by the *multiplier*—that the multiplier *must* be a pure number (i.e. abstract)—that the law of commutation holds, e.g. 3 groups of 4 things is identical with 4 groups of three (the same) things, etc. This knowledge is brought to bear upon the "mystery" of division. For the sake of simplicity let us take our old example: $4 × 3 = $12 =, also, $3 × 4. Now in division there is given either

(*a*) The product $12 and the factor, $4, to find the other factor—3. Or

(*b*) The product $12 and the factor, 3, to find the other factor $4.

In the one case (*a*) we are searching for *times*, i.e. for the *multiplier*, which with the given multiplicand will make the product $12. In the other case (*b*) we are searching for the unit-group $4, the *multiplicand* from which with the given multiplier, 3, the product $12 may be found. In the former case (*a*) the divisor is concrete and the quotient *necessarily* abstract—a pure number. In the latter case (*b*) the divisor is abstract and the quotient necessarily concrete. Speaking somewhat loosely, therefore, we may say that there are two kinds of division: division (*a*) in the sense that a number *contains* a given number a certain (required) number of times, and (*b*) division in the sense that a number ($12) is to be distributed into a given number of unit-groups of required value ($4). But these "two kinds of division" are *not* "widely and radically different." On the contrary they are essentially *correlative:* The one implies the other—the number of unit-groups cannot be found without their *value*—i.e. the number of units in each—nor the *value* of the unit-groups without their number. In both cases the searching operation is precisely the same, and in both is implied the idea of *division into equal parts.* The difference, such as it is, is *not* in the process, or in the fundamental principles which underlie the process; it is in *the interpretion of the result*—in the one case TIMES is meant, in the other case the number of units in the correlative *unit-group.*

For example: $12 ÷ $4 = 3—not "3 *four dollars*"; here the quotient is a pure number, is in fact the *ratio* of $12 to $4? Again: $12 ÷ 4 = $3: here the quotient not only involves the relation (ratio) of 12 to 4; but also *names the standard unit;* in other words it is a concrete number expressing the absolute value of the unit-group. In the first example, we have the answer to the question (as it might be put), What is the *ratio* of $12 to $4. In the second example we have the answer to the question: *four* is the ratio of $12 to what number of dollars?

In view of these fundamental principles, it is indeed surprising that these *two mutually related* processes should be declared fundamentally different; so different, in fact, that different names are needed to mark properly the difference. It is perhaps more surprising that some generally thoughtful men—e.g., the author of the "Philosophy of Arithmetic"—state unconditionally that the divisor can *never* be an abstract number." Let us venture to illustrate the real process in the "two," divisions, working (to make the illustration more inclusive) by partial quotients as in 'long' division.

(*a*) Divide $12 by $4: $12 ÷ $4 = ? i.e. find the *multiplier* which, with $4 for a *multiplicand*, will give $12. Or, still further, using the idea of partial quotients, the question might be stated: $12 = $4 × (1 + 1 + . . .) find the number within the brackets.