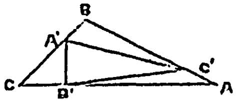

Let A', B', C' be the points of section of BC, CA, AB, respectively.

Then $CA' = \frac{ma}{m+n}$; $CB' = \frac{nb}{m+n}$

$\therefore$ area $CA'B' = \frac{1}{2}\frac{ma}{m+n}\frac{nb}{m+n}\sin C$;

similarly $AB'C' = \frac{1}{2}\frac{mb}{m+n}\frac{nc}{m+n}\sin A$;

similarly $BC'A' = \frac{1}{2}\frac{mc}{m+n}\frac{na}{m+n}\sin B$.

Summing the three areas,

$$\frac{1}{2}\frac{mn}{(m+n)^2}\left\{ab\sin C + bc\sin A + ca\sin B\right\}$$

$$= \frac{1}{2}\frac{mn}{(m+n)^2}\left\{3\,bc\sin A\right\}$$

since $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{bc\sin A + ca\sin B + ab\sin C}{3abc}$; hence

$$\frac{\text{area } ABC}{\text{area } A'B'C'} = \frac{\frac{1}{2}bc\sin A}{\frac{1}{2}bc\sin A - \frac{3}{2}\frac{mn}{(m+n)^2}bc\sin A}$$

$$= \frac{(m+n)^2}{m^2 - mn + n^2}.$$

It follows that the second triangle is to the third in the same ratio, and so on; hence

$$\frac{\text{area } ABC}{\text{area } r^{\text{th}} \text{ inscribed } \triangle} = \left\{\frac{(m+n)^2}{m^2 - mn + n^2}\right\}^r.$$

Also, putting A, A_1, A_2, etc., A_r for the areas of the given triangle ABC and the 1st, 2nd, etc., r^{th} inscribed triangles.

$$\frac{A}{A_1} = \frac{A_2}{A_3} = \frac{A_4}{A_5} = \text{etc.}$$

$$= \frac{A_{2r}}{A_{2r+1}} = \frac{(m+n)^2}{m^2 - mn + n^2};$$

hence

$$\frac{f.\,(A,\ A_2,\ \dots\dots A_{2r})}{f.\,(A_1,\ A_3 \dots\dots A_{2r+1})} = \frac{(m+n)^2}{m^2 - mn + n^2}.$$

138. Sum the infinite series

$$\frac{1}{1.2}\tan\theta + \frac{1}{2.3}\tan^2\theta - \frac{1}{3.4}\tan^3\theta - \frac{1}{4.5}\tan^4\theta + \text{etc.}$$

Write the series,

$$\frac{2-1}{1.2}\tan\theta + \frac{3-2}{2.3}\tan^2\theta - \frac{4-3}{3.4}\tan^3\theta - \frac{5-4}{4.5}\tan^4\theta + + \text{etc.}$$

which readily breaks up into

$\tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \text{etc.}$ $(=\theta)$

$-\frac{1}{2}\tan\theta + \frac{1}{4}\tan^3\theta - \frac{1}{6}\tan^5\theta + \text{etc.}$

$(= \frac{1}{\tan\theta}\log\cos\theta)$

$\frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + \frac{1}{6}\tan^6\theta - \text{etc.}$

$(= -\log\cos\theta)$

$-\frac{1}{3}\tan^2\theta + \frac{1}{5}\tan^4\theta - \frac{1}{7}\tan^6\theta + \text{etc.}$

$(= -\frac{1}{\tan\theta}\left\{\tan\theta - \theta\right\})$

adding the partial sums and simplifying.

Series $=$

$(\cot\theta - 1)\log\cos\theta + (\cot\theta + 1)\,\theta - 1.$

This result may also be arrived at by multiplying the proposed series by $\tan\theta$, differentiating with respect to $\tan\theta$, summing and integrating.

139. Prove that

$$\begin{vmatrix} \frac{b+c}{a}, & \frac{a}{b+c}, & \frac{a}{b+c} \\ \frac{b}{c+a}, & \frac{c+a}{b}, & \frac{b}{c+a} \\ \frac{c}{a+b}, & \frac{c}{a+b}, & \frac{a+b}{c} \end{vmatrix} = \frac{2(a+b+c)^3}{(a+b)(b+c)(c+a)}$$

Reducing each row to a common denominator and dividing thro' by

$$\frac{1}{abc\,(a+b)\,(b+c)\,(c+a)}$$

$$\begin{vmatrix} (b+c)^2, & a^2, & a^2 \\ b^2, & (c+a)^2, & b^2 \\ c^2, & c^2, & (a+b)^2 \end{vmatrix} = 2\,abc\,(a+b+c)^3$$

For a new determinant, subtract the 2nd column from the 1st for the 1st row; subtract the 3rd column from the 2nd for the 2nd row; and add all three columns for the 3rd row.