$$\left.\begin{array}{ll} t\,\Delta_2^{\frac{1}{5}} = a_1\,\Delta_1^{\frac{2}{5}}, & t^2\,a_2\,\Delta_2^{\frac{2}{5}} = h_1\,\Delta_1^{\frac{4}{5}}, \\ t^3\,e_2\,\Delta_2^{\frac{3}{5}} = \Delta_1^{\frac{1}{5}}, & t^4\,h_2\,\Delta_2^{\frac{4}{5}} = e_1\,\Delta_1^{\frac{3}{5}}, \\ \text{therefore } \Delta_2 = a_1^5\,\Delta_1^2, & a_2^5\,\Delta_2^2 = h_1^5\,\Delta_1^4, \\ e_2^5\,\Delta_2^3 = \Delta_1, & h_2^5\,\Delta_2^4 = e_1^5\,\Delta_1^3. \end{array}\right\} \quad (63)$$

Now $a_1^5\,\Delta_1^2$, being equal to $\Delta_2$, is a root of the equation $\psi_1(x) = 0$. And $a_1^5\,\Delta_1^2$, involving only surds that occur in $r_1$, is in a simple state. Therefore, by Prop. III., $a_2^5\,\Delta_2^2$ is a root of the equation $\psi_1(x) = 0$. Therefore $h_1^5\,\Delta_1^4$, and therefore also $h_2^5\,\Delta_2^4$ or $e_1^5\,\Delta_1^3$, are roots of that equation. Hence all the terms

$$\Delta_1,\ a_1^5\,\Delta_1^2,\ e_1^5\,\Delta_1^3,\ h_1^5\,\Delta_1^4, \qquad (64)$$

are roots of the equation $\psi_1(x) = 0$. But $a_1$, $e_1$, $h_1$, are all distinct from zero; for, by (63), if one of them was zero, all would be zero, and therefore $\Delta_1^{\frac{1}{5}}$ would be zero; which by §6, is impossible. From this it follows that no two terms in (64) are equal to one another; for taking $a_1^5\,\Delta_1^2$ and $e_1^5\,\Delta_1^3$, if these were equal, we should have $e_1\,t'\,\Delta_1^{\frac{1}{5}} = a_1$, $t'$ being a fifth root of unity; which; which by §8, is impossible. This gives the equation $\psi_1(x) = 0$ four unequal roots; which, because it is of the second degree, is impossible. Therefore the first term in (55) is not equal to the second in (39). In the same way it can be shown that it is not equal to the third. Therefore it must be equal to the fourth. In like manner the first in (39) is equal to the fourth in (55). Because then $t\,\Delta_2^{\frac{1}{5}} = h_1\,\Delta_1^{\frac{4}{5}}$, and $\Delta_1^{\frac{1}{5}} = t^4\,h_2\,\Delta_2^{\frac{4}{5}}$, $h_2\,\Delta_2 = h_1\,\Delta_1$. But, just as it was proved in §56 that, the roots of the sub-auxiliary $\psi_1(x) = 0$ being the $c$ terms $\Delta_1$, $\Delta_2$, etc., there is no particular cognate form of $E\Delta$ that is not a term in the series $e_1\,\Delta_1$, $e_2\,\Delta_2$, ...., $e_c\,\Delta_c$, it follows that, if $h_1$ be a particular cognate form of $H$, there is no particular cognate form of $H\Delta$ that is not equal to one of the terms $h_1\,\Delta_1$ and $h_2\,\Delta_2$. Hence, since $h_1\,\Delta_1 = h_2\,\Delta_2$, $H\Delta$ has no particular cognate form different in value from $h_1\,\Delta_1$. Therefore, by Prop. III., $h_1\,\Delta_1$ is rational.