UNIVERSITY WORK.

MATHEMATICS.

ARCHIBALD MACMURCHY, M.A., TORONTO, EDITOR.

SOLUTIONS.

SEE NOVEMBER NO.

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1. If O be the centre of the circle described about the triangle ABC and AO, BO, CO, be produced to meet the opposite sides in D, E, F, the circle in D', E', F', respectively, prove that $\frac{DD'}{AD}+\frac{EE'}{BE}+\frac{FF'}{CF}=1$.

1. $\triangle ABC + \triangle D'BC = 2$ quad'l $OBD'C$,
$\triangle ABC + \triangle E'CA = 2$ quad'l $OCE'A$,
$\triangle ABC + \triangle F'AB = 2$ quad'l $OAF'B$,
$\therefore\ 3\triangle ABC+(\triangle D'BC+\triangle E'CA+\triangle F'AB)$
$=2$ hexagon $AF'BD'CE'$
$=2\triangle ABC+2(\triangle D'BC+\triangle E'CA+\triangle F'AB)$,
hence $\triangle ABC=\triangle D'BC+\triangle E'CA+\triangle F'AB$;
again, $\frac{D'D}{DA}=\frac{\triangle D'BC}{\triangle ABC}$, etc.,

$$\therefore\ \frac{D'D}{DA}+\frac{E'E}{EB}+\frac{F'F}{FC}=\frac{\triangle D'BC+\triangle E'CA+\triangle F'AB}{\triangle ABC}=1.$$

2. If $x+y+z=x^{-1}+y^{-1}+z^{-1}=0$, prove $\frac{x^6+y^6+z^6}{x^3+y^3+z^3}=\frac{x^9+y^9+z^9}{x^6+y^6+z^6}=xyz$, and $x^5+y^5+z^5=0$.

2. Since $x+y+z=0$, and $xy+yz+zx=0$, $\therefore\ x^3+y^3+z^3=3xyz$, and $x^3y^3+y^3z^3+z^3x^3=3x^2y^2z^2$.
Again, $x^6+y^6+z^6=(x^3+y^3+z^3)^2-2(x^3y^3+y^3z^3+z^3x^3)=9x^2y^2z^2-6x^2y^2z^2=3x^2y^2z^2$
and $x^9+y^9+z^9=(x^3+y^3+z^3)^3-3(x^3+y^3+z^3)(x^3y^3+y^3z^3+z^3x^3)+3x^3y^3z^3$
$=27x^3y^3z^3-27x^3y^3z^3+3x^3y^3z^3=3x^3y^3z^3$,

$$\therefore\ \frac{x^6+y^6+z^6}{x^3+y^3+z^3}=\frac{3x^2y^2z^2}{3xyz}=xyz,$$

and $\frac{x^9+y^9+z^9}{x^6+y^6+z^6}=\frac{3x^3y^3z^3}{3x^2y^2z^2}=xyz$.

3. If $x=bz+cy$
$y=cx+az$
$z=ay+bx$,
prove $\frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}=\frac{z^2}{1-c^2}$,

$$\frac{\sqrt{1-a^2}}{a}+\frac{\sqrt{1-b^2}}{b}+\frac{\sqrt{1-c^2}}{c}=\frac{\sqrt{1-a^2}\,.\,\sqrt{1-b^2}\,.\,\sqrt{1-c^2}}{abc}$$

3. (1) $x^2=bzx+cxy$
$y^2=cxy+ayz$,
$\therefore\ x^2-y^2=z(bx-ay)=(bx+ay)(bx-ay)=b^2x^2-a^2y^2$
$\therefore\ x^2(1-b^2)=y^2(1-a^2)$, or $\frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}$.
Similarly $\frac{x^2}{1-a^2}=\frac{z^2}{1-c^2}$,

$$\therefore\ \frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}=\frac{z^2}{1-c^2}$$

(2) $\frac{x}{yz}=\frac{b}{y}+\frac{c}{z}$ and $\frac{y}{zx}=\frac{c}{z}+\frac{a}{x}$,

$\therefore$ multiplying $\frac{1}{z^2}=\frac{ab}{xy}+\frac{c}{z}\left(\frac{a}{x}+\frac{b}{y}\right)+\frac{c^2}{z^2}$

$$\therefore\ \frac{1-c^2}{z^2}=\frac{ab}{xy}+\frac{bc}{yz}+\frac{ca}{zx}=\frac{abc}{xyz}\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)$$

and $\frac{1-c^2}{c^2}=\frac{z^2}{c^2}\cdot\frac{abc}{xyz}\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)$

$$\therefore\ \frac{\sqrt{1-c^2}}{c}+\frac{\sqrt{1-a^2}}{a}+\frac{\sqrt{1-b^2}}{b}$$

$$=\left\{\frac{abc}{xyz}\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\right\}^{\frac{1}{2}}\left(\frac{z}{c}+\frac{x}{a}+\frac{y}{b}\right)$$

$$=\frac{xyz}{abc}\left\{\frac{abc}{xyz}\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\right\}^{\frac{3}{2}}$$