5. Find three numbers in geometrical progression such that their sum shall be 21, and the sum of their squares 189.

6. Define the trigonometrical ratios of an angle less than 90°, and prove:

(1) $\sin^2 A + \cos^2 A = 1$

(2) $\sin A \cos A = \dfrac{1}{\tan A + \cot A}$

7. Prove the following formulæ:

(1) $\sin \overline{A-B} = \sin A \cos B - \cos A \sin B.$

(2) $\tan \frac{1}{2} A = \dfrac{1-\cos A}{\sin A}$

8. In any triangle establish the following relations:

(1) $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

(2) $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

(3) Area $= \sqrt{s(s-a)(s-b)(s-c)}$

9. Having given two sides and the included angle of a triangle, obtain formulæ from which to find the other two angles and the third side.

10. Discuss the ambiguous case in the solution of triangles.

11. Find the sine and cosine of 45° and 30°, and deduce those of 75° and 15°.

SOLUTIONS.

1. (1) $\dfrac{x^2}{y^2} = \dfrac{a^2}{b^2}$ $\therefore \dfrac{x^2}{x^2+y^2} = \dfrac{a^2}{a^2+b^2} = \dfrac{x^2}{c^2}$ $\therefore$ $x = \pm \dfrac{ac}{\sqrt{(a^2+b^2)}}$ &c.

(2)

$$\begin{array}{r|l}
A = & 2 \quad 4 - 3 - \ 22 = 0 \\
B = & 4 - 2 \quad 5 - \ 18 = 0 \\
C = & 6 \quad 3 - 2 - \ 31 = 0 \\
\hline
A+B+C & 12 \quad 5 \quad 0 - \ 71 = 0 = D \\
\hline
3C - 2A = & 14 \quad 1 \quad 0 - \ 49 = 0 = E \\
5E - D & 58 \quad 0 \quad 0 - 174 = 0
\end{array}$$

or $58x - 174 = 0$ $x=3$, $y=7$, $z=4$.

See McLellan's *Handbook*, page 178.

2. (1) Multiply 2nd by 2; add and subtract; take square roots and $x+y=\pm 9$, $x-y=\pm 1$, $\therefore x=\pm 5$ or $y=\pm 4$.

(2) $1+6-15=-4-4$ $\therefore$ $x+1$ is a factor.

See McLellan's *Handbook*, page 42, § 16.

$$\begin{array}{r|l|l}
 & 1 - 4 + \ 6 - \ 4 & -15 \\
-1 & \ \ -1 + \ 5 - 11 & +15 \\
\hline
 & 1 - 5 + 11 & -15 \\
+3 & \ \ +3 - \ 6 & +15 \\
\hline
 & 1 - 2 + \ 5 &
\end{array}$$

Now the roots of $x^2-2x+5=0$, are $1 \pm 2\sqrt{-1}$, which with -1 and 3 are all the roots of the $=n$.

(3) 2nd ÷ 1st $= x^2 - xy + y^2 = 3$. Combine with 1st and $xy=2$, $\therefore$ from 1st $x+y=\pm 3$, and from 2nd $x-y=\pm 1$, $\therefore$ &c.

3. Book-work. (1) do.

(2) $a=3$ and $a+6d=16$, $\therefore d=\frac{13}{6}$, and the series is $3, 5\frac{1}{6}, 7\frac{1}{3}, 9\frac{1}{2}, 11\frac{2}{3}, 13\frac{5}{6}, 16.$

4. (1) Book-work.

(2) $\dfrac{a}{1-r} = 3\frac{1}{3}$, and $ar = -\frac{5}{2}$, $\therefore$ $r = +\frac{3}{2}$ or $-\frac{1}{2}$.

Now the latter value only will apply, $\because r<1$, $\therefore a=5$, and the series is $5 - \frac{5}{2} + \frac{5}{4} - \frac{5}{8} +$ &c.

5. $a+ar+ar^2=21$, and $a^2+a^2r^2+a^2r^4=189$.

Dividing, $a-ar+ar^2=9$, $\therefore$ from 1st $ar=6$, or $a=\dfrac{6}{r}$

Substituting $\dfrac{6}{r}+6+6r=21$, *i.e.* $6r^2-15r+6=0$,

Or $(2r-1)(r-2)=0$, $\therefore r=\frac{1}{2}$ or 2, and $a=12$ or 3.

Hence the numbers are 12, 6, 3, or 3, 6, 12.

6. (1) Book work.

(2) $\dfrac{1}{\tan A + \cot A} = \dfrac{1}{\dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}}$ &c.

7. (1) Book-work.

(2) $\tan \frac{1}{2} A = \dfrac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \dfrac{1-\cos A}{\sin A}$.

8, 9, 10, 11. Book-work

ALGEBRA AND TRIGONOMETRY.

HONORS.

Examiner—W. FITZGERALD, M.A.

1. Solve,

(1) $\begin{cases} x^2+xy=65 \\ y^2-xy=24 \end{cases}$

(2) $\begin{cases} x^3+y^3+(x+y)xy=13 \\ \dfrac{x^2y^2}{x+y}=36 \end{cases}$

2. Find the number of variations of n different letters taken r together; also the number of such variations, when each may enter 1, 2, 3, &c., or r times in each variation.

If the number of variations of $a+b$ things taken two together be 56, and of $a-b$ things 12, find the number of combinations of a things, taken b together.

3. State the Binomial Theorem, and prove it when the index is a positive integer.

Expand to five terms, $(a-3x)^{-\frac{1}{3}}$.

4. Find the present value of an annuity A for n years at compound interest.

The reversion of a freehold estate worth P pounds per annum to commence a years hence is to be sold. Ascertain its present value at r per cent. per annum compound interest.

5. Define a *continued fraction*; and illustrate the method of converting a quadratic surd to a continued fraction.

Express as continued fractions

(1) $\sqrt{11}$; (2) $\sqrt{13}$; (2) $\sqrt{17}$.

6. What is a recurring series?

Explain what is meant by the *scale of relation* of a recurring series.

Sum to n terms, and *ad infinitum* the series

$$\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \text{\&c.}$$

7. Find the radii of the inscribed and escribed circles of a triangle in terms of the sides and angles.

8. In any triangle prove:

(1) $\dfrac{\sin(B-C)}{\sin(C-A)} = \dfrac{(b^2-c^2)\sin B}{(c^2-a^2)\sin A}$

(2) Area $= \frac{1}{2}(b^2+c^2)\dfrac{a \sin B \sin C}{b \sin B + c \sin C}$.

9. Show how to expand a^x in a series of ascending powers of x.

10. State Demoivre's Theorem, and assuming its truth prove,

(1) $\cos a = 1 - \dfrac{a^2}{1\cdot2} + \dfrac{a^4}{1\cdot2\cdot3\cdot4} \ldots$ &c.

(2) $\sin a = a - \dfrac{a^3}{1\cdot2\cdot3} +$ &c....

11. Sum to n terms:

$$\sin\theta - \sin(\theta+a) + \sin(\theta+2a) \ldots$$

and deduce the sum of n terms of the series, $\cos\theta - \cos 2\theta + \cos 3\theta$ ....&c.

SOLUTIONS.

1. (1) Put $y=vx$, and we have $x^2+vx^2=65$, and $v^2x^2-vx^2=24$. Divide these equals and cancel and $(1+v)\div(v^2-v)=\frac{65}{24}$. Whence $65v^2-89v-24=0$, and $v=\frac{8}{5}$, or $-\frac{3}{13}$, Whence x and $y=$&c.

(2) (By W. N. Watson, Seaforth.)

Transform first equation into $(x^2+y^2)(x+y)=13$ and multiply by second and we have $(x^2+y^2)x^2y^2=13\times2^2\times3^2$. Now, looking at the form of each side we see that they correspond and that $x^2+y^2=13$, $x^2=2^2$, and $y^2=3^2$ will satisfy the equation. Whence $x=\pm2$. $y=\mp3$. Then by dividing down the equation with these values we get a quadratic which will give the other two values of x and of y.

2. (a) $V_n = n(n-1)(n-2)\ldots(n-r+1)$.

(b) Let a be placed before each of the n things a, b, c, d, &c., thus forming n variations. Similarly for b, c, d, &c., each of the rest. Thus there will be formed n sets with n variations in each set when letter enters twice in each variation, *i.e.* $n\times n$ or n^2 variations altogether. Again place a before each of these n^2 variations, and form n^2 variations 3 and 3 together. Place b, c, d, &c., &c. The whole number of variations is n^r.