$$R_1^{\frac{1}{n}} = A_1(P_m^m \phi_\sigma^\sigma \; \ldots \; F_\beta^\beta)^{\frac{1}{n}},$$
$$R_z^{\frac{1}{n}} = A_z(P_{zm}^m \phi_{z\sigma}^\sigma \; \ldots \; F_{z\beta}^\beta)^{\frac{1}{n}},$$
$$R_{ez}^{\frac{1}{n}} = A_{ez}(P_{ezm}^m \phi_{ez\sigma}^\sigma \ldots \; F_{ez\beta}^\beta)^{\frac{1}{n}}.$$

Therefore
$$\left.\begin{aligned}(R_z R_1^{-z})^{\frac{1}{n}} &= (A_z A_1^{-z})\,(P_{zm} P_m^{-z})^{\frac{m}{n}}\,(\phi_{z\sigma}\phi_\sigma^{-z})^{\frac{\sigma}{m}} \ldots \\ \text{and} \quad (R_{ez} R_e^{-z})^{\frac{1}{n}} &= (A_{ez} A_e^{-z})(P_{ezm} P_{em}^{-z})^{\frac{m}{n}}(\phi_{ez\sigma}\phi_{e\sigma}^{-z})^{\frac{\sigma}{n}} \ldots\end{aligned}\right\} \qquad (136)$$

Because $(P_{zm}P_m^{-z})^{\frac{m}{n}}$ and other such expressions have been shown to be rational functions of the primitive n^{th} root of unity, the two equations (106) correspond respectively to (3) and (5). If z be not prime to n, and yet not a multiple of n, it may be taken to be ev, where v is equal to $\frac{n}{y}$, y being one of the terms in the series (107) distinct from n, and w^e being the general primitive n^{th} root of unity. Then, just as we obtained the pair of equations (136) by means of (109), we can now, by means of (110), obtain

$$\left.\begin{aligned}(R_{ev} R_1^{-ev})^{\frac{1}{n}} &= (A_{ev} A_1^{-ev})\,(P_{evm} P_m^{-ev})^{\frac{m}{n}} \ldots \\ (R_{cev} R_c^{-ev})^{\frac{1}{n}} &= (A_{cev} A_c^{-ev})(P_{cevm} P_{cm}^{-ev})^{\frac{m}{n}} \ldots\end{aligned}\right\} \qquad (137)$$

where w^c represents any one of the primitive n^{th} roots of unity. Because $(P_{evm}P_m^{-ev})^{\frac{m}{n}}$ and other such expressions have been shown to be rational functions of the primitive n^{th} root of unity, the two equations (137) correspond respectively to (3) and (5). Finally, should z be a multiple of n, it may be taken to be zero. Then the equation corresponding to (3) is

$$R_z^{\frac{1}{n}} = q_1 R^{\frac{z}{n}},$$

q_1 being a rational function of w. Or, since $z = 0$,

$$R_0^{\frac{1}{n}} = q_1.$$

But $R_0^{\frac{1}{n}}$ is rational. Therefore q_1 is rational. Hence, if q_e be what q_1 becomes in passing from w to w^e, $q_e = q_1$. Also $R_{ez}^{\frac{1}{n}} = R_0^{\frac{1}{n}} = q_e$. Therefore, since $R_e^{\frac{z}{n}} = 1$,

$$R_{ez}^{\frac{1}{n}} = q_e R_e^{\frac{z}{n}},$$

which is the equation corresponding to (5). Therefore, whatever z be, the equation (5) subsists along with (3). Hence, by the Criterion in §10, the expression (105) is the root of a pure uni-serial Abelian equation of the n^{th} degree.